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REAL LIFE APPLICATIONS OF TIGHT FACE IRREGULARITY STRENGTH UNDER WHEEL GRAPHS INCLUDING VERTICES, EDGES AND FACES

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Abstract

Wheel graphs can be constructed by adding an additional vertex on the center of a cycle graph and then joining this vertex with all other surrounding vertices. A k –labeling of type (α,β,γ) in a wheel graph is a labeling from graph elements into the set of positive integers. Face irregularity strength of a wheel graph W_n with respect to k –labeling of type (α,β,γ) is the minimum integer k on which the face weights of the graph are distinct. We investigate the exact value for the face irregularity strength with respect to vertex k –labeling, edge k –labeling and total k –labeling of wheel graphs. Some real life applications on face irregularity strength of graphs are discovered and their models are discussed which will be helpful in practical situations.

Keywords: Wheel graphs, Face irregularity strength, Applications of irregularity strength of graphs.

INTRODUCTION

All graphs in this research are simple, finite, plane and undirected. Planar graphs with vertex set V, edge set E and face set F can be demonstrated as $G = (V,E,F)$. Let $\alpha,\beta,\gamma \in \{0,1\}$ and k is a positive integer, then a graph k –labeling under a mapping Ψ of type (α,β,γ) is a labeling from graph elements (V,E,F) into the set of positive integers$\{1,2,3,\ldots,k\}$. Graph labeling always depends on domain of labeling. If the domain of labeling is the vertex set of graph, then labeling will be named as vertex k−labeling of type (1, 0, 0). In case, the domain of labeling is edge set, face set, vertex-edge set, edge-face set, vertex-face set or vertex-edge-face set, then their corresponding names will be edge k –labeling of type$(0,1,0)$, face k –labeling of type$(0,0,1)$, total k –labeling of type $(1,1,0)$, edge-face k –labeling of type $(0,1,1)$, vertex-face k –labeling of type $(1,0,1)$ or entire k –labeling of type $(1,1,1)$ respectively. The labeling, under discussion in this article is total k –labeling of type$(1,1,0)$. Reader can go through [18] for further details on k –labeling of planer graphs.

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The origin of graph theory is associated with Euler, who investigated a smart path among seven bridges in 1736, known as Eulerian circuit which is a trail that starts and ends on the same vertex. The initial work on graph theory can be studied in $[1, 2, 3, 4, 6]$. Graph theory has many applications in everyday life like, hyperlinks connect web pages in computer networking system, connection between cities and countries through trains and aeroplanes are well defined applications of graph theory. Cities or countries play their roles as vertices and the path among them are edges. The size of the monitor screen which is measured in inches and centimeters is directly related to the labeling of a cycle graph with four vertices. The shape of a pizza is a wheel graph which can be sliced into different faces of the wheel graphs. The multiple television channels running on a screen of a single monitor is a clear example of face labeling of a grid graph. When Alice talks with Bob on phone then they both act like vertices and the connection between them behaves like an edge. If we add Kate, Sara and Roma in their discussion then they all make a graph with five vertices. To get an approach on daily life graph theory, we recommend reader to study $[10, 13]$.

Face labeling of graphs under k −labeling of type (α, β, γ) has been recently introduced in graph theory $[14]$. Later, face irregularity strength under k −labeling of graphs of type $(1, 1, 0)$ was published by Aleem et al. $[18]$. Face labeling of graphs under k −labeling of type (α, β, γ) by using its particular types of vertices, edges and faces can be explained to the readers by following figure1.

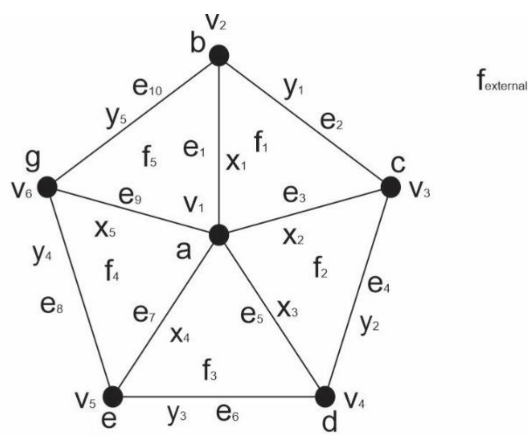

figure 1
Wheel Graph W6

Remember that in k −labeling of type (α, β, γ), the labels can not be 0 for any vertex, edge or face and the minimum label must be 1. In figure 1, v_1, v_2, v_3, v_4, v_5 and v_6 are vertices of the graph and f_1, f_2, f_3, f_4, f_5 and $f_{External}$ are internal and external faces of the graph, the external face represents all the area outside the graph. The alphabets a, b, c, d, e and g are the labels of vertices of the graph, the alphabets x_1, x_2, x_3, x_4, x_5 and y_1, y_2, y_3, y_4, y_5 are the labels of internal and external edges of the graph respectively. When working on face irregularity strength of graphs under k −labeling of type (α, β, γ), the weights are always calculated by adding labels of surrounding vertices and edges under some conditions. Mathematically, weight can be represented by the following equation:

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$$W_{\Psi(\alpha,\beta,\gamma)}(f) = \alpha \sum_{v\sim f} \Psi(v) + \beta \sum_{e\sim f} \Psi(e) + \gamma\Psi(f)$$

Where k –labeling Ψ of type (α,β,γ) of any plane graph G is called face irregular k –labeling of type (α,β,γ) if any two distinct faces $f, g \in G$ have their distinct weights, that is, $W_{\Psi(\alpha,\beta,\gamma)}(f) \neq W_{\Psi(\alpha,\beta,\gamma)}(g)$. Face irregularity strength of type (α,β,γ) of any plane graph G is the least positive integer k for which the graph has face irregular k –labeling of type (α,β,γ).

Note that every addition in these graphs will be commutative. In figure 1, weight of vertex v_1 will be calculated by adding label of this particular vertex and labels of its surrounding edges. So, weight of vertex $v_1 = a + x_1 + x_2 + x_3 + x_4 + x_5$. Weights of other vertices can be calculated in the similar way.

In figure 1, weight of edge can be calculated by adding label of this particular edge and labels of its surrounding vertices. So, weight of edge $e_1 = x_1 + a + b$. Weights of other edges can be calculated in the similar way.

In figure 1, the weight of the face 1 can be calculated by adding label of face 1 and labels of all other surrounding vertices and edges. So, weight of face $1 = a + b + c + x_1 + x_2 + y_1$. Weights of other faces can be calculated in the similar way.

All the labels which are used in figure 1 are positive integers except zero. The label which will be minimum from the set of positive integers and maximum among all the assigned labels will be the face irregularity strength of this graph. Irregularity strength of graphs, vertex labeling, edge labeling and face labeling can be studied further in $[9, 12, 8, 5, 7, 11, 16, 17, 18]$.

Section 2 contains investigation of face irregularity strength of wheel graphs under k –labeling of type (α,β,γ). Section 3 includes real life application of face irregularity strength in computer networking. Section 4 is based on conclusions of this research.

Face Irregularity Strength of Wheel Graphs

Let us begin now the calculation of face irregularity strength of wheel graphs under k –labeling of type (α,β,γ). The vertex set and edge set for wheel graphs can be written as

$V\,(Wn) = \{vi\,;\, i = 1, 2, 3, \ldots, n\}$.

$E(Wn) = \{v_1v_i\,;\, i = 2, 3, \ldots, n - 1\} \cup \{v_iv_{i+1}\,;\, i = 2, 3, \ldots, n - 1\} \cup \{v_nv_2\}$.

Theorem 1 Let G be a wheel graph with n vertices including the center vertex v_1 which is connected to all other vertices v_i of G for $2 \leq i \leq n$ and $n \geq 5$. Then face irregularity strength of G with respect to vertex k –labeling Ψ of type (α,β,γ) will be $n - 1$, that is,

$VFS(G) = n - 1$.

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Proof. Suppose that $G = W_n$ represents a wheel graph with n vertices. In general, for a wheel vertex k –labeling of type $(1, 0, 0)$, we have

$$\Psi(v_i) = \begin{cases} 1, & i = 1, \\ \left\lceil \frac{i-1}{2} \right\rceil, & i = 2,3,4, \dots, n-1, \\ n-1, & i = n. \end{cases}$$

And face weights are

$$\Psi(f_i) = \begin{cases} 2+i, & i = 1,2,3, \dots, n-3, \\ 2 + (n-3) + \left\lceil \frac{n}{2} \right\rceil, & i = n-2, \\ 2 + (n-3) + 2, & i = n-1. \end{cases}$$

We prove this theorem by mathematical induction.

For$n = 5$, we have

$$\Psi(v_i) = \begin{cases} 1, & i = 1, \\ \left\lceil \frac{i-1}{2} \right\rceil, & i = 2,3,4, \\ 4, & i = 5. \end{cases}$$

The weight of external face will be 10 and the weights of four internal faces will become

$$\Psi(f_i) = \begin{cases} 2+i, & i = 1,2, \\ 7, & i = 3, \\ 6, & i = 4. \end{cases}$$

Since all the weights are distinct. So, by the definition of face irregularity strength of graphs, the results are true for $n = 5$. Suppose that the results are true for $n = k$. Then the vertices and weights will become

$$\Psi(v_i) = \begin{cases} 1, & i = 1, \\ \left\lceil \frac{i-1}{2} \right\rceil, & i = 2,3,4, \dots, k-1, \\ k-1, & i = k. \end{cases}$$

And face weights are

$$\Psi(f_i) = \begin{cases} 2+i, & i = 1,2,3, \dots, k-3, \\ 2 + (k-3) + \left\lceil \frac{k}{2} \right\rceil, & i = k-2, \\ 2 + (k-3) + 2, & i = k-1. \end{cases}$$

We have assumed that the results are true for $1 \leq vi \leq k$. Now if we increase the vertices of graph then graph labels and graph weights will change accordingly. Every face of wheel graph is a construction by three edges so it means that one edge of face 1 will also be an edge of face 2. Similarly one edge of face 2 will also be an edge of face 3 and vice versa. Hence vertices v_k and v_{k+1} will have one common edge which indicates that both v_k and v_{k+1} follow same pattern of labeling. Mathematically, by using 1 and 2, we can follow the pattern for $\{1, 2, 3, \dots, k-1, k, k+1, \dots, n\}$. Hence, we have

$$\Psi(v_i) = \begin{cases} 1, & i = 1, \\ \left\lceil \frac{i-1}{2} \right\rceil, & i = 2,3,4,\dots,k, \\ k+1-1, & i = k+1. \end{cases}$$

Also, we have

$$\Psi(v_i) = \begin{cases} 1, & i = 1, \\ \left\lceil \frac{i-1}{2} \right\rceil, & i = 2,3,4,\dots,k,k+1,\dots,n-1, \\ n-1, & i = n. \end{cases}$$

The face weights will be as follows

$$\Psi(f_i) = \begin{cases} 2+i, & i = 1,2,3,\dots,n-3, \\ 2+(n-3)+\left\lceil \frac{n}{2} \right\rceil, & i = n-2, \\ 2+(n-3)+2, & i = n-1. \end{cases}$$

Since all the face weights are distinct so by the definition of face irregularity strength of graphs, the results are true for $k+1$. This completes the proof.

Theorem 2. Let G be a wheel graph with n vertices including the center vertex v_1 which is connected to all other vertices v_i of G for $2 \leq i \leq n$ and $n \geq 5$. Then face irregularity strength of G with respect to edge k –labeling Ψ of type (α,β,γ) will be $n-2$ and $n-3$ for even and odd values of n respectively, that is,

$$EFS\ (W_n) = \begin{cases} n-2, & if\ n\ is\ even, \\ n-3, & if\ n\ is\ odd. \end{cases}$$

Proof. Let W_n is a wheel graph with n vertices. The general estimations for the edge k –labeling of type $(0,1,0)$ of a wheel graph can be written as

Case 1: When n is even

$$\Psi(v_1v_i) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4,\dots,n-1, \\ n-2; & i = n. \end{cases}$$

$$\Psi(v_iv_{i+1}) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4,\dots,n-1, \\ n-2; & i = n. \end{cases}$$

$$\Psi(v_nv_2) = n-2$$

Case 2: When n is odd

$$\Psi(v_1v_i) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4,\dots,n-1, \\ n-3; & i = n. \end{cases}$$

$$\Psi(v_iv_{i+1}) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4,\dots,n-1, \\ n-2; & i = n. \end{cases}$$

$$\Psi(v_nv_2) = n-3$$

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And face weights are

$$\Psi(f_i) = \begin{cases} 2+i; & i = 1,2 \\ Wf_{(i-2)}+3; & i = 3,5,7,\dots,n-4, \\ Wf_{(i-2)}+3; & i = 4,6,8,\dots,n-3, \\ Wf_{(n-3)}+\left\lceil \frac{n}{2} \right\rceil - 1; & i = n-2, \\ Wf_{(n-2)}-1; & i = n-1 \; where \; n \; is \; odd \\ Wf_{(n-2)}+1; & i = n-1 \; where \; n \; is \; even \end{cases}$$

We prove this theorem by mathematical induction.

For $n = 5$, the case 2 will provide the following results for labeling.

Case 2: When n is odd

$$\Psi(v_1v_i) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4, \\ 2; & i = 5. \end{cases}$$

$$\Psi(v_iv_{i+1}) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4, \\ 3; & i = 5. \end{cases}$$

$$\Psi(v_nv_2) = 2$$

And the corresponding four internal face weights can be calculated as

$$\Psi(f_i) = \begin{cases} 2+i; & i = 1,2 \\ 6; & i = 3, \\ 5; & i = 4, \end{cases}$$

Since all the face weights are distinct so by the definition of face irregularity strength of graphs, the results are true for $n = 5$. If we talk about even value of n, then we will take $n = 6$. So for $n = 6$, the case 1 will provide following results for labeling.

Case 1: When n is even

$$\Psi(v_1v_i) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4,5, \\ 4; & i = 6. \end{cases}$$

$$\Psi(v_iv_{i+1}) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4,5, \\ 4; & i = 6. \end{cases}$$

$$\Psi(v_nv_2) = 4,$$

And the corresponding face weights are

$$\Psi(f_i) = \begin{cases} 2+i; & i = 1,2, \\ 6; & i = 3, \\ 8; & i = 4, \\ 9; & i = 5, \end{cases}$$

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Since all the face weights are distinct so by the definition of face irregularity strength of graphs, the results are true for $n = 6$. Hence, condition 1 is satisfied for both even and odd cases. Suppose that the results are true for $n = k$. Then we need to discuss both even and odd values of n. First suppose that n is even then we have

Case 1: When n is even

$$\Psi(v_1 v_i) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4,\dots,k-1, \\ k-2; & i = k. \end{cases}$$

$$\Psi(v_i v_{i+1}) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4,\dots,k-1, \\ k-2; & i = k. \end{cases}$$

$$\Psi(v_n v_2) = k-2,$$

And the face weights are

$$\Psi(f_i) = \begin{cases} 2+i; & i = 1,2, \\ Wf_{(i-2)} + 3; & i = 3,5,7,\dots,k-4, \\ Wf_{(i-2)} + 3; & i = 4,6,8,\dots,k-3, \\ Wf_{(k-3)} + \left\lceil \frac{k}{2} \right\rceil - 1; & i = k-2, \\ Wf_{(k-2)} + 1; & i = k-1 \ where\ k\ is\ even \end{cases}$$

Now we consider the case when $n = k$ is odd, so we have

Case 2: When n is odd

$$\Psi(v_1 v_i) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4,\dots,k-1, \\ k-3; & i = k. \end{cases}$$

$$\Psi(v_i v_{i+1}) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4,\dots,k-1, \\ k-2; & i = k. \end{cases}$$

$$\Psi(v_n v_2) = k-3,$$

And the face weights are

$$\Psi(f_i) = \begin{cases} 2+i; & i = 1,2, \\ Wf_{(i-2)} + 3; & i = 3,5,7,\dots,k-4, \\ Wf_{(i-2)} + 3; & i = 4,6,8,\dots,k-3, \\ Wf_{(k-3)} + \left\lceil \frac{k}{2} \right\rceil - 1; & i = k-2, \\ Wf_{(k-2)} - 1; & i = k-1 \ where\ k\ is\ odd \end{cases}$$

Note that wheel graphs are connected graphs where every face is a combination of three edges, in which one edge is a common edge between every two faces. By our supposition that the results are true for $n = k$, we mean that all results are true from $1 \leq vi \leq k$. It means that increasing the labels from k to $k + 1$ will also affect the weights. So if we talk

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about pattern of changing from k to $k + 1$ then it should be same for k and $k + 1$. Hence by using $3 - 10$, the results can be defined for $k + 1$ as well by assuming the sequence $\{1, 2, 3, \ldots, k - 1, k, k + 1, \ldots, n\}$. So, we have the following even and odd cases.

Case 1: When n is even

$$\Psi(v_1 v_i) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4, \ldots, (k+1) - 1, \\ (k+1) - 2; & i = k+1. \end{cases}$$

$$\Psi(v_i v_{i+1}) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4, \ldots, (k+1) - 1, \\ (k+1) - 2; & i = k+1. \end{cases}$$

$$\Psi(v_n v_2) = (k+1) - 2,$$

And the face weights are

$$\Psi(f_i) = \begin{cases} 2 + i; & i = 1,2, \\ Wf_{(i-2)} + 3; & i = 3,5,7, \ldots, (k+1) - 4, \\ Wf_{(i-2)} + 3; & i = 4,6,8, \ldots, (k+1) - 3, \\ Wf_{((k+1)-3)} + \left\lceil \frac{k+1}{2} \right\rceil - 1; & i = (k+1) - 2, \\ Wf_{((k+1)-2)} + 1; & i = (k+1) - 1 \ where\ k\ is\ even \end{cases}$$

Now we consider the case when $n = k$ is odd, so we have

Case 2: When n is odd

$$\Psi(v_1 v_i) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4, \ldots, (k+1) - 1, \\ (k+1) - 3; & i = k+1. \end{cases}$$

$$\Psi(v_i v_{i+1}) = \begin{cases} \left\lceil \frac{i-1}{2} \right\rceil; & i = 2,3,4, \ldots, (k+1) - 1, \\ (k+1) - 2; & i = k+1. \end{cases}$$

$$\Psi(v_n v_2) = (k+1) - 3,$$

And the face weights are

$$\Psi(f_i) = \begin{cases} 2 + i; & i = 1,2, \\ Wf_{(i-2)} + 3; & i = 3,5,7, \ldots, (k+1) - 4, \\ Wf_{(i-2)} + 3; & i = 4,6,8, \ldots, (k+1) - 3, \\ Wf_{((k+1)-3)} + \left\lceil \frac{k+1}{2} \right\rceil - 1; & i = (k+1) - 2, \\ Wf_{((k+1)-2)} - 1; & i = (k+1) - 1 \ where\ k\ is\ odd \end{cases}$$

In both of the above cases, for even and odd, the weights are distinct. So, by the definition of face irregularity strength of graphs, the results are true for $k + 1$. So

$$Efs(W_n) = \begin{cases} n - 2 & if\ n\ is\ even \\ n - 3; & if\ n\ is\ odd \end{cases}$$

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Theorem 3. Let G be a wheel graph with n vertices including the center vertex v_1 which is connected to all other vertices v_i of G for $2 \leq i \leq n$ and $n \geq 5$. Then face irregularity strength of G with respect to total k –labeling Ψ of type (α, β, γ) will be $n - 2$.

Proof. The general estimations for the total k –labeling of type $(1,1,0)$ of wheel graph can be written as

$$\begin{cases} \Psi(v_1 v_i) = 1; \quad i = 2,3,4,\dots,n-1, \\ \{\Psi(v_i v_{i+1}) = 1\} \cup \{\Psi(v_n v_2) = 1\}; \quad i = 2,3,4,\dots,n-1. \end{cases}$$

$$\Psi(v_i) = \begin{cases} 1; \quad i = 1(center\ vertex), \\ \left\lceil \frac{i-1}{2} \right\rceil; \quad i = 2,3,4,\dots,n-1, \\ n-2; \quad i = n. \end{cases}$$

And the face weights can be calculated as,

$$\Psi(f_i) = \begin{cases} 5 + i; \quad i = 1,2,3,\dots,n-3, \\ 5 + (n-3) + \left\lceil \frac{n}{2} \right\rceil - 1; \quad i = n-2, \\ 5 + (n-3) + 1; \quad i = n-1 \end{cases}$$

The following figure 2 represents the total labeling of type $(1,1,0)$.

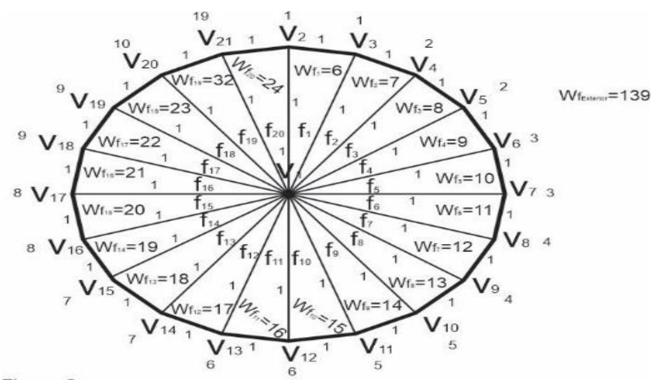

Figure 2
Face Irregularity Strength of W_n with respect to Total 19-Labeling of type (1,1,0)

When we talk about the face irregularity strength of W_n then note that it can not be found directly from W_n. We need to go through small graphs in order to approach W_n. Note that n can not be zero. We prove this theorem by contradiction. Let us assume W_{21}, the wheel graph with 21 vertices, as you can see in figure 2. Suppose, on contrary, that the face irregularity strength of W_{21} is not 19. In figure 2, we start labeling form face 1, then labeling of face 2, face 3 and then so on till face 20. Now if face irregularity strength k of W_{21} is not 19 then the possible values are $1 \leq k < 19$ or $19 < k$. Note that weights should be distinct but labels can be repeated. Suppose on contrary that face irregularity strength of W_{21} is 1 then if we replace label 19 with 1 then $Wf_{20} = 6$, but we see that $Wf_1 = 6$ so there is a repetition in face weights. Hence by the definition of face irregularity strength of graphs, the face weights should be distinct. So our supposition is wrong for $k = 1$ and so 1 is not the face irregularity strength of W_{21}. Now let us assume some other aspect that we have to avoid repetition in labels to get the exact face irregularity strength. So

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assume that $k = 11$, then by replacing 19 with 11, we get $Wf_{20} = 15$ which is again a repetition in face weights. Hence some other aspect is also wrong because we again got a contradiction. Now we conclude that $k \neq \{1, 2, 3, \dots, 17\}$. Now suppose that $k = 18$ then $Wf_{20} = 22$ which is again a contradiction. So now the only possible value that we can assume further is 20. Replacing 19 with 20, we get $Wf_{20} = 25$ which is a new weight so we got a number 20 on which the face weights of W_{21} are distinct. But now we have a new problem that by the definition of face irregularity strength, the number k should be the least positive integer and the least positive integer between 19 and 20 is 19. Hence there is a contradiction on our supposition. Hence the only possible value for face irregularity strength of W_{21} is 19. So, we conclude that

$TFS(W_n) = n - 2 \; for \; all \; values \; of \; n$.

Applications

Figure 3 shows a computer networking model which is applicable to both, Local Area Network and Wide Area Network. where one server computer (PC 1) is attached with thirteen other client computers (PC 2 - PC 14). Note that PC denotes personal computer. The numbers $1, 2, 3, 4, 5, 6$ and 12 are the labels of vertices and edges. The alphabets f_1, f_2, f_3 and so on f_{13} are representing face 1, face 2, face 3 and so on face 13 respectively. The numbers inside the box, starting from 6 and ending on 16 are the weights of faces. These can be calculated by adding all the labels around any particular face.

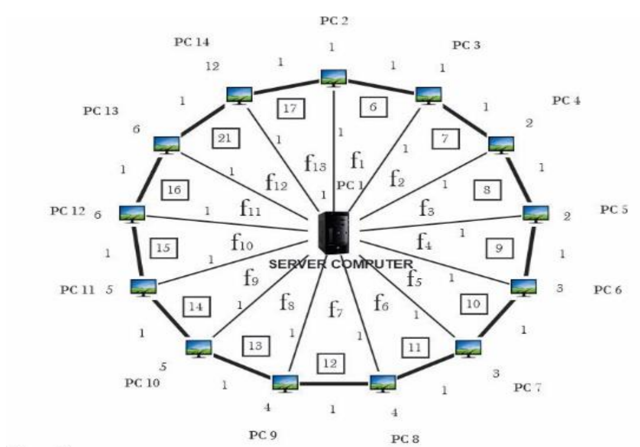

Figure 3
Computer Networking Plan for LAN and WAN based on Face Irregularity Strength

Label of PC 2 will be used as the code to open PC 2 and to access PC 2 by server. Similarly, labels of PC 3, PC 4 and so on till PC 14 will be used to open themselves and by the server. This model may be named as equal rights model. The label of server will be used as a password to open server computer. The label on the edge between server and PC 2 will be used as a data sharing code for both, server computer and PC 2. Similarly, the labels on edges between {PC 2, PC 3, ..., PC 14} and server will be used to share data among them. Edge labeling in graph theory provides strong applications in everyday life. The edge labels in between PC 2 and PC 3 will be used to share information from PC 2 to PC 3 and also from PC 3 to PC 2. Similarly the edges among all other PCs

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will work on same strategy. We observe one thing that PC 1, PC 2 and PC 3 make a cycle graph C_3 and this cycle contains a face f_1. Weight of this face can be calculated by adding all the labels around this face, that is, $Wf1 = 1 + 1 + 1 + 1 + 1 + 1 = 6$. Similarly other face weights can be measured by the similar method. These face weights will be used as a code to access PCs in these faces, from other PCs existed in other faces. For example, if server is busy or not responding or shut down then other PCs will keep working on regular basis by the help of these weights. The face which has label of face irregularity strength will be named as primary face. Primary face totally depends on discipline of labeling. There are many types of labelings that can be applied on wheel graphs. Even if we change the style of k –labeling then there might be some other face which includes face irregularity strength. The face irregularity strength of this network is 12. In our research, the primary face can have full rights like a server and after server, primary face will have right to convert other PCs into server. Face irregularity strength lies in two faces, that is face 12 and face 13.

$Wf_{12} = 21$,

$Wf_{13} = 17$,

Since $17 < 21$, so we will choose face 13 as the primary face. Now all the PCs which are attached with face 13 have equal rights like server. Suppose that server is shut down, not working, broken or facing corrupted windows or facing any other issues, then all the PCs in face 13 will act as servers. Now suppose that PC 6 in face 10 wants to install LATEX software but PC 6 cannot install anything without having full rights. So, it can request PC 14 or PC 2 in face 13 (face with irregularity strength) to assign full rights. Then PC 6 will be eligible to install any software. So now PC2, PC 14 and PC 6 have equal rights but PC 6 cannot assign server rights to any other PC. Similarly suppose that PC 10 wants to install VLC player to watch English movie 'Avatar'. PC 10 will ask PC 2 or PC 14 to assign full rights. Then PC 10 can install VLC but note that PC 10 cannot give permission to any other PC to have full rights. So basically three systems in a network can work as server but all systems in a network can have full rights on computers. Face weights of all computers can be displayed on them so that every other system can use that code to send message to that particular system. This 'Equal Rights Network System' can be helpful in making right PC networks in the field of computer science. This type of computer network system is designed first time by the help of graph theory so improvements can be discussed with the passage of time.

CONCLUSION

In this research, authors worked on calculation of face irregularity strength of wheel graphs which are an extension of cycle graphs. Face labeling of graphs have many applications in our real life. Some applications have been discussed in this research article. There are many more applications which can be related to edge k –lableing of type $(0, 1, 0)$ because an edge represents a relation between any two objects and everything is this universe is related to at least one other thing by a strong edge so edge relationship has a wide application in the field of graph theory.

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