

SOME FIXED POINT RESULTS ON A VECTOR VALUED S -METRIC SPACE

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ABSTRACT

We establish the existence of fixed point results on a vector S -metric space which is complete. The concept of S -metric space was introduced by Sedghi et al. [10] in 2012. By combining the concept of vector metric space and S -metric space, the notion of vector S -metric space is introduced. We also give some examples to authenticate our results.

Keywords: Vector lattice, Vector metric space, Vector S -metric space.

1 Introduction

Fixed point theory gives an important tool for proving the existence and uniqueness of the solutions. Yao and Yang [5] pointed out that every S -metric space is b -metric space. Several authors like Sedghi et al. [10], Kim et al.[4], Shahraki et al. [8], Özgür et al. [9] proved fixed point results on S -metric space. Altun and Cevik [1] defined vector metric spaces in 2009. We hereby define vector S -metric space which is Riesz space valued. In this paper, we establish fixed point results on vector S -metric space.

Definition 1.1 [6] Let $k: H \rightarrow H$ be a map. Then k is said to have a fixed point $\alpha \in H$ if $k(\alpha) = \alpha$.

Remark 1.2 A map may have no fixed point, unique fixed point or several fixed points.

Definition 1.3[7] On a set C , a relation \leq is a partial order if it follows the conditions stated below:

- (a) $b_1 \leq b_1$ (reflexive)
 - (b) $b_1 \leq b_2$ and $b_2 \leq b_1$ implies $b_1 = b_2$ (anti-symmetry)
 - (c) $b_1 \leq b_2$ and $b_2 \leq b_3$ implies $b_1 \leq b_3$ (transitivity)
- $\forall b_1, b_2, b_3 \in C$. The pair (C, \leq) is known as partially ordered set.

A partially ordered set (C, \leq) is called linearly ordered if for $b_1, b_2 \in C$, we have either $b_1 \leq b_2$ or $b_2 \leq b_1$.

Definition 1.4[2] Let C be linear space which is real and (C, \leq) be a poset. Then the poset (C, \leq) is said to be an ordered linear space if it follows the properties mentioned below:

- (a) $b_1 \leq b_2 \Rightarrow b_1 + b_3 \leq b_2 + b_3$
- (b) $b_1 \leq b_2 \Rightarrow \omega b_1 \leq \omega b_2 \quad \forall b_1, b_2, b_3 \in C \text{ and } \omega > 0.$

Definition 1.5[2] A poset is called lattice if each set with two elements has an infimum and a supremum.

Definition 1.6 [2] If each subset has infimum and supremum then lattice is complete.

Definition 1.7[2] An ordered linear space where the ordering is lattice is called vector lattice. This is also called Riesz space.

Example 1.8[2] Let \mathbb{R}^b ($b \geq 1$) be the vector space of b tuples $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_b)$ and $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_b)$ with coordinatewise multiplication and addition. If we define $\vartheta \leq \vartheta$ means $\vartheta_\alpha \leq \vartheta_\alpha$ holds for $1 \leq \alpha \leq b$, then \mathbb{R}^b is a vector lattice w.r.t. partial ordering.

Definition 1.9[2] Let V be vector lattice with non-negative cone $V^+ = \{\vartheta \in V : \vartheta \geq 0\}$ for an element $\vartheta \in V$, the negative part ϑ^- , the positive part ϑ^+ and the absolute part $|\vartheta|$ are denoted as

$$\vartheta^- = (-\vartheta) \vee 0, \vartheta^+ = \vartheta \vee 0, |\vartheta| = \vartheta \vee (-\vartheta).$$

Also $|\vartheta| = 0$ iff $\vartheta = 0$.

Definition 1.10[2] A vector lattice V is called Archimedean if $\inf\{\frac{1}{m}\vartheta\} = 0$ for every $\vartheta \in V^+$ where

$$V^+ = \{\vartheta \in V : \vartheta \geq 0\}.$$

Definition 1.11[1] Let V be a vector lattice and \mathfrak{R} be a nonvoid set. Then vector metric is a mapping $d: \mathfrak{R} \times \mathfrak{R} \rightarrow V$ on \mathfrak{R} if it follows the conditions stated below:

- (a) $d(b_1, b_2) = 0$ iff $b_1 = b_2$

(b) $d(b_1, b_2) \leq d(b_1, b_3) + d(b_3, b_2) \forall b_1, b_2, b_3 \in \mathfrak{R}$

The triplet (\mathfrak{R}, d, V) is called vector metric space.

Definition 1.12[8] Let \mathfrak{R} be a nonvoid set. Then S -metric is a mapping $S: \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \rightarrow [0, \infty)$ on \mathfrak{R} if it follows the conditions stated below:

(a) $S(b_1, b_2, b_3) \geq 0$,

(b) $S(b_1, b_2, b_3) = 0$ iff $b_1 = b_2 = b_3$,

(c) $S(b_1, b_2, b_3) \leq S(b_1, b_2, \alpha) + S(b_2, b_3, \alpha) + S(b_3, b_1, \alpha)$,

for all $b_1, b_2, b_3, \alpha \in \mathfrak{R}$.

Then (\mathfrak{R}, S) is called S -metric space .

Now, we define vector S -metric space as follows:

Definition 1.13 Let V be a vector lattice and \mathfrak{R} be a nonvoid set. Then vector S -metric is a mapping $S: \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \rightarrow V$ on \mathfrak{R} if it satisfies the conditions mentioned below:

(a) $S(b_1, b_2, b_3) \geq 0$,

(b) $S(b_1, b_2, b_3) = 0$ iff $b_1 = b_2 = b_3$,

(c) $S(b_1, b_2, b_3) \leq S(b_1, b_2, \alpha) + S(b_2, b_3, \alpha) + S(b_3, b_1, \alpha)$,

for all $b_1, b_2, b_3, \alpha \in \mathfrak{R}$.

The triplet (\mathfrak{R}, S, V) is called vector S -metric space.

Example 1.14 Let \mathfrak{R} be a nonvoid set and V be a vector lattice. If a function $S: V \times V \times V \rightarrow V$ defined by

$$S(b_1, b_2, b_3) = |(b_1 - b_3)| + |(b_3 - b_2)| + |(b_2 - b_1)| \quad \forall b_1, b_2, b_3 \in \mathfrak{R}$$

then the triplet (\mathfrak{R}, S, V) is a vector S -metric space.

Definition 1.15 In vector S -metric space (\mathfrak{R}, S, V) , a sequence $\langle \vartheta_n \rangle \in \mathfrak{R}$ is called V -convergent to some $\vartheta \in \mathfrak{R}$ if there is a sequence $\langle \mu_n \rangle$ in V satisfying $\mu_n \downarrow 0$ and $S(\vartheta_n, \vartheta_n, \vartheta) \leq \mu_n$ and denote it by $\vartheta_n \xrightarrow{S,V} \vartheta$.

Definition 1.16 In vector S -metric space (\mathfrak{R}, S, V) , a sequence $\langle \vartheta_n \rangle \in \mathfrak{R}$ is known as V -Cauchy sequence if $\exists \langle \mu_n \rangle \in V$ satisfying $\mu_n \downarrow 0$ and $S(\vartheta_n, \vartheta_n, \vartheta_{n+q}) \leq \mu_n \forall q$ and n .

Definition 1.17 A vector S -metric space (\mathfrak{R}, S, V) is called V -complete if all V -Cauchy sequence in \mathfrak{R} is V -convergent to a limit in \mathfrak{R} .

Definition 1.18 Let (\mathfrak{R}, S, V) and (\mathfrak{R}', S', V') be two vector S -metric spaces. A function $K: (\mathfrak{R}, S, V) \rightarrow (\mathfrak{R}', S', V')$ is continuous at $\alpha \in \mathfrak{R}$ if $\exists \langle h_b \rangle \in \mathfrak{R}$ with $h_b \xrightarrow{S,V} \alpha$ then $K(h_b) \xrightarrow{S',V'} K(\alpha)$.

Lemma 1.19 If (\mathfrak{R}, S, V) is a symmetric vector S -metric space, then prove that

$$S(\vartheta, \vartheta, \mu) = S(\mu, \mu, \vartheta) \quad \forall \mu, \vartheta \in \mathfrak{R}.$$

Proof. Let $\mu, \vartheta \in \mathfrak{R}$, then

$$\begin{aligned} S(\vartheta, \vartheta, \mu) &\leq S(\vartheta, \vartheta, \vartheta) + S(\vartheta, \vartheta, \vartheta) + S(\mu, \mu, \vartheta) \\ &= S(\mu, \mu, \vartheta) \end{aligned} \tag{1}$$

$$\begin{aligned} S(\mu, \mu, \vartheta) &\leq S(\mu, \mu, \mu) + S(\mu, \mu, \mu) + S(\vartheta, \vartheta, \mu) \\ &= S(\vartheta, \vartheta, \mu) \end{aligned} \tag{2}$$

By (1) and (2), we get $S(\vartheta, \vartheta, \mu) = S(\mu, \mu, \vartheta)$.

Lemma 1.20 Let (\mathfrak{R}, S, V) be a vector S -metric space. If \exists two sequences $\langle h_b \rangle, \langle \vartheta_b \rangle \in \mathfrak{R}$ such that

$$h_b \xrightarrow{S,V} h \quad \text{and} \quad \vartheta_b \xrightarrow{S,V} \vartheta$$

then prove that

$$\lim_{b \rightarrow \infty} S(h_b, h_b, \vartheta_b) = S(h, h, \vartheta)$$

Proof. Since

$$\lim_{b \rightarrow \infty} h_b = h \quad \text{and} \quad \lim_{b \rightarrow \infty} \vartheta_b = \vartheta$$

then $\exists \langle \lambda_n \rangle, \langle \mu_n \rangle \in V$ satisfying $\lambda_n \downarrow 0, \mu_n \downarrow 0$ and

$$\begin{aligned} S(h_b, h_b, h) &\leq \lambda_n, \\ S(\vartheta_b, \vartheta_b, \vartheta) &\leq \mu_n \quad \forall b \text{ and } n \end{aligned}$$

then

$$\begin{aligned}
 S(\hbar_b, \hbar_b, \vartheta_b) &\leq 2S(\hbar_b, \hbar_b, \hbar) + S(\vartheta_b, \vartheta_b, \hbar) \\
 &\leq 2S(\hbar_b, \hbar_b, \hbar) + 2S(\vartheta_b, \vartheta_b, \vartheta) + S(\hbar, \hbar, \vartheta) \\
 &\leq 2\lambda_n + 2\mu_n + S(\hbar, \hbar, \vartheta) \\
 S(\hbar_b, \hbar_b, \vartheta_b) - S(\hbar, \hbar, \vartheta) &\leq 2\lambda_n + 2\mu_n
 \end{aligned} \tag{3}$$

Also

$$\begin{aligned}
 S(\hbar, \hbar, \vartheta) &\leq 2S(\hbar, \hbar, \hbar_b) + S(\vartheta, \vartheta, \hbar_b) \\
 &\leq 2S(\hbar, \hbar, \hbar_b) + 2S(\vartheta, \vartheta, \vartheta_b) + S(\hbar_b, \hbar_b, \vartheta_b) \\
 &= 2S(\hbar_b, \hbar_b, \hbar) + 2S(\vartheta_b, \vartheta_b, \vartheta) + S(\hbar_b, \hbar_b, \vartheta_b) \\
 &\leq 2\lambda_n + 2\mu_n + S(\hbar_b, \hbar_b, \vartheta_b) \\
 S(\hbar, \hbar, \vartheta) - S(\hbar_b, \hbar_b, \vartheta_b) &\leq 2\lambda_n + 2\mu_n
 \end{aligned} \tag{4}$$

So, by relation (3) and (4), by using $|\vartheta| = \vartheta \vee (-\vartheta)$ we get

$$|S(\hbar_b, \hbar_b, \vartheta_b) - S(\hbar, \hbar, \vartheta)| \leq 2\lambda_n + 2\mu_n$$

Since $\lambda_n \downarrow 0$ and $\mu_n \downarrow 0$ and $|\vartheta| = 0$ iff $\vartheta = 0$ this implies

$$\lim_{b \rightarrow \infty} S(\hbar_b, \hbar_b, \vartheta_b) = S(\hbar, \hbar, \vartheta)$$

Lemma 1.21 Let (\mathfrak{R}, S, V) be a vector S -metric space. If \exists two sequences $\langle \hbar_b \rangle, \langle \vartheta_b \rangle \in \mathfrak{R}$ and $\exists \langle \eta_b \rangle \in V$ satisfying $\langle \eta_b \rangle \downarrow 0$ such that

$$S(\hbar_b, \hbar_b, \vartheta_b) \leq \eta_b \quad \text{for all } b$$

whenever $\langle \hbar_b \rangle \in \mathfrak{R}$ and

$$\hbar_b \xrightarrow{S,V} \hbar \quad \text{for some } \hbar \in \mathfrak{R}$$

then

$$\vartheta_b \xrightarrow{S,V} \hbar.$$

Proof. We have

$$S(\vartheta_b, \vartheta_b, \hbar) \leq 2S(\vartheta_b, \vartheta_b, \hbar_b) + S(\hbar, \hbar, \hbar_b)$$

Since

$$\hbar_b \xrightarrow{S,V} \hbar$$

then $\exists \langle \mu_b \rangle \in V$ satisfying $\langle \mu_b \rangle \downarrow 0$ and $S(\hbar_b, \hbar_b, \hbar) \leq \mu_b$ for all b .

Then

$$S(\vartheta_b, \vartheta_b, \hbar) \leq 2S(\vartheta_b, \vartheta_b, \hbar_b) + S(\hbar_b, \hbar_b, \hbar) \leq 2\eta_b + \mu_n$$

Hence

$$\vartheta_b \xrightarrow{S,V} \hbar.$$

Remark 1.22 [3] If V be a vector lattice and $b \leq \lambda b$ where $b \in V^+$, $\lambda \in [0,1)$, then $b = 0$.

2 Main Results

Here we establish some results on vector S -metric space that extend some of the results of [7], [11], [5] and [4].

Lemma 2.1 Let (\mathfrak{R}, S, V) be a vector S -metric space which is complete and V -Archimedean. Let a sequence $\langle \hbar_b \rangle$ be in \mathfrak{R} such that

$$S(\hbar_b, \hbar_b, \hbar_{b+1}) \leq \alpha S(\hbar_{b-1}, \hbar_{b-1}, \hbar_b) \quad \forall b \in \mathbb{N}$$

where $\alpha \in [0,1)$. Then $\langle \hbar_b \rangle$ is a V -Cauchy sequence in \mathfrak{R} .

Proof. Using (1), we get

$$S(\hbar_b, \hbar_b, \hbar_{b+1}) \leq \alpha S(\hbar_{b-1}, \hbar_{b-1}, \hbar_b) \leq \alpha^2 S(\hbar_{b-2}, \hbar_{b-2}, \hbar_{b-1}) \leq \dots \leq \alpha^b S(\hbar_0, \hbar_0, \hbar_1)$$

So, for $b > \ell$, we have

$$\begin{aligned} S(\hbar_\ell, \hbar_\ell, \hbar_b) &\leq 2S(\hbar_\ell, \hbar_\ell, \hbar_{\ell+1}) + S(\hbar_b, \hbar_b, \hbar_{\ell+1}) \\ &= 2S(\hbar_\ell, \hbar_\ell, \hbar_{\ell+1}) + S(\hbar_{\ell+1}, \hbar_{\ell+1}, \hbar_b) \\ &\leq 2S(\hbar_\ell, \hbar_\ell, \hbar_{\ell+1}) + 2S(\hbar_{\ell+1}, \hbar_{\ell+1}, \hbar_{\ell+2}) + S(\hbar_b, \hbar_b, \hbar_{\ell+2}) \\ &= 2S(\hbar_\ell, \hbar_\ell, \hbar_{\ell+1}) + 2S(\hbar_{\ell+1}, \hbar_{\ell+1}, \hbar_{\ell+2}) + S(\hbar_{\ell+2}, \hbar_{\ell+2}, \hbar_b) \\ &\leq 2S(\hbar_\ell, \hbar_\ell, \hbar_{\ell+1}) + 2S(\hbar_{\ell+1}, \hbar_{\ell+1}, \hbar_{\ell+2}) + \dots + S(\hbar_{b-1}, \hbar_{b-1}, \hbar_b) \\ &< 2S(\hbar_\ell, \hbar_\ell, \hbar_{\ell+1}) + 2S(\hbar_{\ell+1}, \hbar_{\ell+1}, \hbar_{\ell+2}) + \dots + 2S(\hbar_{b-1}, \hbar_{b-1}, \hbar_b) \\ &< 2(\alpha^\ell + \alpha^{\ell+1} + \dots + \alpha^{b-1})S(\hbar_0, \hbar_0, \hbar_1) \\ &< 2\alpha^\ell(1 + \alpha + \alpha^2 + \dots)S(\hbar_0, \hbar_0, \hbar_1) \\ &< 2\frac{\alpha^\ell}{1-\alpha}S(\hbar_0, \hbar_0, \hbar_1) \downarrow 0 \quad \ell \rightarrow \infty. \end{aligned}$$

Thus $\langle \hbar_b \rangle$ is a V -Cauchy sequence in \mathfrak{R} .

Theorem 2.2 Let (\mathfrak{R}, S, V) be a vector S -metric space which is complete and V -Archimedean. Let $k: \mathfrak{R} \rightarrow \mathfrak{R}$ be a map such that

$$S(k\hbar, k\vartheta, k\mu) \leq qU(\hbar, \vartheta, \mu) \quad 0 \leq q < \frac{1}{3} \quad \text{and} \quad \hbar, \vartheta, \mu \in \mathfrak{R}$$

where

$$\begin{aligned} U(\hbar, \vartheta, \mu) \in \{ & S(\hbar, \vartheta, \mu), S(\hbar, \hbar, k\hbar), S(\vartheta, \vartheta, k\vartheta), S(\mu, \mu, k\mu), \\ & S(\hbar, \hbar, k\vartheta), S(\vartheta, \vartheta, k\mu), S(\mu, \mu, k\hbar) \} \end{aligned}$$

Then k has a unique fixed point.

Proof. Assume that $\sigma_0 \in \mathfrak{R}$ and

$$\sigma_b = k\sigma_{b-1} \geq 1.$$

Then

$$\begin{aligned} S(\sigma_b, \sigma_b, \sigma_{b+1}) &= S(k\sigma_{b-1}, k\sigma_{b-1}, k\sigma_b) \\ &\leq qU(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) \end{aligned}$$

where

$$\begin{aligned} U(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) \in \{ & S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b), S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_{b-1}), S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_{b-1}), \\ & S(\sigma_b, \sigma_b, k\sigma_b), S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_{b-1}), S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_b), \\ & S(\sigma_b, \sigma_b, k\sigma_{b-1}) \} \\ = \{ & S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b), S(\sigma_b, \sigma_b, \sigma_{b+1}), S(\sigma_{b-1}, \sigma_{b-1}, \sigma_{b+1}), 0 \} \end{aligned}$$

The possible cases are:

$$(i) S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq qS(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$$

$$(ii) S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq qS(\sigma_b, \sigma_b, \sigma_{b+1})$$

and so

$$S(\sigma_b, \sigma_b, \sigma_{b+1}) = 0.$$

$$(iii) S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq qS(\sigma_{b-1}, \sigma_{b-1}, \sigma_{b+1})$$

$$\leq q[2S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_{b+1}, \sigma_{b+1}, \sigma_b)]$$

$$= q[2S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_b, \sigma_b, \sigma_{b+1})]$$

$$S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \frac{2q}{(1-q)} S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$$

(iv) $S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq q \cdot 0 = 0$

and so

$$S(\sigma_b, \sigma_b, \sigma_{b+1}) = 0.$$

Thus $S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \lambda S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$ where $\lambda \in \{q, \frac{2q}{(1-q)}\} < 1$

Since V is Archimedean, by lemma (2.1) $\langle \sigma_b \rangle$ is a V -Cauchy sequence in \mathfrak{R} and \mathfrak{R} is V -complete, $\exists \omega \in \mathfrak{R}$ such that $\sigma_b \xrightarrow{S,V} \omega$. Hence there exist $\langle \alpha_b \rangle \in V$ such that $\alpha_b \downarrow 0$ and

$$S(\sigma_b, \sigma_b, \omega) \leq \alpha_b.$$

Now, we show that ω is a fixed point of k . Since

$$\begin{aligned} S(\omega, \omega, k\omega) &\leq 2S(\omega, \omega, k\sigma_b) + S(k\omega, k\omega, k\sigma_b) \\ &= 2S(\omega, \omega, \sigma_{b+1}) + qU(\omega, \omega, \sigma_b) \end{aligned} \tag{5}$$

where

$$\begin{aligned} U(\omega, \omega, \sigma_b) &\in \{S(\omega, \omega, \sigma_b), S(\omega, \omega, k\omega), S(\omega, \omega, k\omega), S(\sigma_b, \sigma_b, k\sigma_b), \\ &\quad S(\omega, \omega, k\omega), S(\omega, \omega, k\sigma_b), S(\sigma_b, \sigma_b, k\omega)\} \\ &= \{S(\omega, \omega, \sigma_b), S(\omega, \omega, k\omega), S(\sigma_b, \sigma_b, \sigma_{b+1}), S(\omega, \omega, \sigma_{b+1}), \\ &\quad S(\sigma_b, \sigma_b, k\omega)\} \end{aligned}$$

We have the following cases:

$$\begin{aligned} (i) S(\omega, \omega, k\omega) &\leq 2S(\omega, \omega, \sigma_{b+1}) + qS(\omega, \omega, \sigma_b) \\ &= 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + qS(\sigma_b, \sigma_b, \omega) \\ &\leq 2\alpha_{b+1} + q\alpha_b \\ &\leq (2+q)\alpha_b \end{aligned}$$

$$(ii) S(\omega, \omega, k\omega) \leq 2S(\omega, \omega, \sigma_{b+1}) + qS(\omega, \omega, k\omega)$$

$$(1-q)S(\omega, \omega, k\omega) = 2S(\sigma_{b+1}, \sigma_{b+1}, \omega)$$

$$S(\omega, \omega, k\omega) \leq \frac{2}{1-q} \alpha_{b+1}$$

$$\leq \frac{2}{1-q} \alpha_b$$

$$\begin{aligned}
 (iii) S(\omega, \omega, k\omega) &\leq 2S(\omega, \omega, \sigma_{b+1}) + qS(\sigma_b, \sigma_b, \sigma_{b+1}) \\
 &\leq 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + q[2S(\sigma_b, \sigma_b, \omega) + S(\sigma_{b+1}, \sigma_{b+1}, \omega)] \\
 &= (2+q)S(\sigma_{b+1}, \sigma_{b+1}, \omega) + 2qS(\sigma_b, \sigma_b, \omega) \\
 &\leq (2+q)\alpha_{b+1} + 2q\alpha_b \\
 &\leq (2+q)\alpha_b + 2q\alpha_b \\
 &= (2+3q)\alpha_b
 \end{aligned}$$

$$\begin{aligned}
 (iv) S(\omega, \omega, k\omega) &\leq 2S(\omega, \omega, \sigma_{b+1}) + qS(\omega, \omega, \sigma_{b+1}) \\
 &= 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + qS(\sigma_{b+1}, \sigma_{b+1}, \omega) \\
 &= (2+q)S(\sigma_{b+1}, \sigma_{b+1}, \omega) \\
 &\leq (2+q)\alpha_{b+1} \\
 &\leq (2+q)\alpha_b
 \end{aligned}$$

$$\begin{aligned}
 (v) S(\omega, \omega, k\omega) &\leq 2S(\omega, \omega, \sigma_{b+1}) + qS(\sigma_b, \sigma_b, k\omega) \\
 &\leq 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + q[2S(\sigma_b, \sigma_b, \omega) + S(k\omega, k\omega, \omega)] \\
 &= 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + q[2S(\sigma_b, \sigma_b, \omega) + S(\omega, \omega, k\omega)]
 \end{aligned}$$

$$(1-q)S(\omega, \omega, k\omega) = 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + 2qS(\sigma_b, \sigma_b, \omega)$$

$$(1-q)S(\omega, \omega, k\omega) \leq 2\alpha_{b+1} + 2q\alpha_b$$

$$(1-q)S(\omega, \omega, k\omega) \leq 2\alpha_b + 2q\alpha_b$$

$$S(\omega, \omega, k\omega) \leq \frac{2(1+q)}{1-q} \alpha_b$$

In the last inequality of each case, the infimum on the right hand side is 0. So

$$S(\omega, \omega, k\omega) = 0$$

that is

$$k\omega = \omega$$

Thus k has a fixed point ω . Now we prove ω is unique. If k has another fixed point ω_1 , then

$$k\omega_1 = \omega_1.$$

Now

$$S(\omega, \omega, \omega_1) = S(k\omega, k\omega, k\omega_1) \leq qU(\omega, \omega, \omega_1)$$

where

$$\begin{aligned} U(\omega, \omega, \omega_1) &\in \{S(\omega, \omega, \omega_1), S(\omega, \omega, k\omega), S(\omega, \omega, k\omega), S(\omega_1, \omega_1, k\omega_1), \\ &S(\omega, \omega, k\omega), S(\omega, \omega, k\omega_1), S(\omega_1, \omega_1, k\omega)\} \\ &= \{S(\omega, \omega, \omega_1), 0\} \end{aligned}$$

This implies

$$S(\omega, \omega, \omega_1) = 0.$$

So $\omega = \omega_1$. Hence k has a unique fixed point ω .

Theorem 2.3 Let (\mathfrak{R}, S, V) be a vector S -metric space which is complete and V -Archimedean. Let $k: \mathfrak{R} \rightarrow \mathfrak{R}$ be a map such that

$$S(k\hbar, k\vartheta, k\mu) \leq qU(\hbar, \vartheta, \mu) \quad 0 \leq q < \frac{1}{2} \quad \text{and} \quad \hbar, \vartheta, \mu \in \mathfrak{R}$$

where

$$\begin{aligned} U(\hbar, \vartheta, \mu) &\in \left\{ \frac{1}{2}[S(\hbar, \hbar, k\hbar) + S(\vartheta, \vartheta, k\vartheta) + S(\mu, \mu, k\mu)], \right. \\ &\frac{1}{2}[S(\hbar, \hbar, k\vartheta) + S(\vartheta, \vartheta, k\mu) + S(\mu, \mu, k\hbar)], \\ &\left. \frac{1}{2}[S(\hbar, \hbar, k\mu) + S(\vartheta, \vartheta, k\hbar) + S(\mu, \mu, k\vartheta)] \right\} \end{aligned}$$

Then k has a unique fixed point.

Proof. Assume that $\sigma_0 \in \mathfrak{R}$ and

$$\sigma_b = k\sigma_{b-1} \geq 1.$$

Then

$$\begin{aligned} S(\sigma_b, \sigma_b, \sigma_{b+1}) &= S(k\sigma_{b-1}, k\sigma_{b-1}, k\sigma_b) \\ &\leq qU(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) \end{aligned}$$

where

$$\begin{aligned}
 U(\sigma_{b-1}, \sigma_b, \sigma_b) &\in \left\{ \frac{1}{2} [S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_{b-1}) + S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_{b-1}) + S(\sigma_b, \sigma_b, k\sigma_b)], \right. \\
 &\quad \frac{1}{2} [S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_{b-1}) + S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_b) + S(\sigma_b, \sigma_b, k\sigma_{b-1})], \\
 &\quad \frac{1}{2} [S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_b) + S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_{b-1}) + S(\sigma_b, \sigma_b, k\sigma_{b-1})] \} \\
 &= \left\{ \frac{1}{2} [S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_b, \sigma_b, \sigma_{b+1})], \right. \\
 &\quad \frac{1}{2} [S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_{b-1}, \sigma_{b-1}, \sigma_{b+1}) + S(\sigma_b, \sigma_b, \sigma_b)], \\
 &\quad \frac{1}{2} [S(\sigma_{b-1}, \sigma_{b-1}, \sigma_{b+1}) + S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_b, \sigma_b, \sigma_b)] \} \\
 &= \left\{ \frac{1}{2} [2S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_b, \sigma_b, \sigma_{b+1})], \frac{1}{2} [S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + \right. \\
 &\quad \left. S(\sigma_{b-1}, \sigma_{b-1}, \sigma_{b+1})] \}
 \end{aligned}$$

The possible cases are:

$$(i) S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \frac{q}{2} [2S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_b, \sigma_b, \sigma_{b+1})]$$

$$(1 - \frac{q}{2})S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq qS(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$$

$$S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \frac{2q}{2-q} S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$$

$$(ii) S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \frac{q}{2} [S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_{b-1}, \sigma_{b-1}, \sigma_{b+1})]$$

$$\leq \frac{q}{2} [S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + 2S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_{b+1}, \sigma_{b+1}, \sigma_b)]$$

$$= \frac{q}{2} [S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + 2S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_b, \sigma_b, \sigma_{b+1})]$$

$$(1 - \frac{q}{2})S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \frac{3q}{2} S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$$

$$S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \frac{3q}{2-q} S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$$

Combining both cases, we get

$$S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \lambda S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$$

where

$$\lambda \in \left\{ \frac{2q}{2-q}, \frac{3q}{2-q} \right\} < 1.$$

Since V is Archimedean, by lemma (2.1) $\langle \sigma_b \rangle$ is a V -Cauchy sequence in \mathfrak{R} and \mathfrak{R} is V -complete, $\exists \omega \in \mathfrak{R}$ such that $\sigma_b \xrightarrow{S,V} \omega$. Hence there exist $\langle \alpha_b \rangle \in V$ such that $\alpha_b \downarrow 0$ and

$$S(\sigma_b, \sigma_b, \omega) \leq \alpha_b.$$

Now, we show that ω is a fixed point of k . Since

$$\begin{aligned} S(\omega, \omega, k\omega) &\leq 2S(\omega, \omega, k\sigma_b) + S(k\omega, k\omega, k\sigma_b) \\ &= 2S(\omega, \omega, \sigma_{b+1}) + qU(\omega, \omega, \sigma_b) \end{aligned} \quad (6)$$

where

$$U(\omega, \omega, \sigma_b) \in \left\{ \frac{1}{2} [S(\omega, \omega, k\omega) + S(\omega, \omega, k\omega) + S(\sigma_b, \sigma_b, k\sigma_b)], \right.$$

$$\frac{1}{2} [S(\omega, \omega, k\omega) + S(\omega, \omega, K\sigma_b) + S(\sigma_b, \sigma_b, k\omega)],$$

$$\left. \frac{1}{2} [S(\omega, \omega, k\sigma_b) + S(\omega, \omega, k\omega) + S(\sigma_b, \sigma_b, k\omega)] \right\}$$

$$\in \left\{ \frac{1}{2} [S(\omega, \omega, k\omega) + S(\omega, \omega, k\omega) + S(\sigma_b, \sigma_b, k\sigma_b)], \right.$$

$$\frac{1}{2} [S(\omega, \omega, k\omega) + S(\omega, \omega, k\sigma_b) + S(\sigma_b, \sigma_b, k\omega)] \}$$

$$U(\omega, \omega, \sigma_b) \in \left\{ \frac{1}{2} [2S(\omega, \omega, k\omega) + S(\sigma_b, \sigma_b, \sigma_{b+1})], \frac{1}{2} [S(\omega, \omega, k\omega) \right.$$

$$\left. + S(\omega, \omega, \sigma_{b+1}) + S(\sigma_b, \sigma_b, k\omega)] \right\}$$

We have the following cases:

$$\begin{aligned} \text{(i)} S(\omega, \omega, k\omega) &\leq 2S(\omega, \omega, \sigma_{b+1}) + \frac{q}{2} [2S(\omega, \omega, k\omega) + S(\sigma_b, \sigma_b, \sigma_{b+1})] \\ &\leq 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + \frac{q}{2} [2S(\omega, \omega, k\omega) + 2S(\sigma_b, \sigma_b, \omega) \\ &\quad + S(\sigma_{b+1}, \sigma_{b+1}, \omega)] \end{aligned}$$

$$(1-q)S(\omega, \omega, k\omega) \leq (2 + \frac{q}{2})S(\sigma_{b+1}, \sigma_{b+1}, \omega) + qS(\sigma_b, \sigma_b, \omega)$$

$$S(\omega, \omega, k\omega) \leq \frac{4+q}{2(1-q)}\alpha_{b+1} + \frac{2q}{2(1-q)}\alpha_b$$

$$\leq (\frac{4+q}{2(1-q)} + \frac{2q}{2(1-q)})\alpha_b$$

$$S(\omega, \omega, k\omega) \leq (\frac{4+3q}{2(1-q)})\alpha_b$$

$$(ii) S(\omega, \omega, k\omega) \leq 2S(\omega, \omega, \sigma_{b+1}) + \frac{q}{2}[S(\omega, \omega, k\omega) + S(\omega, \omega, \sigma_{b+1})]$$

$$+ S(\sigma_b, \sigma_b, k\omega)]$$

$$\leq 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + \frac{q}{2}[S(\omega, \omega, k\omega) + S(\sigma_{b+1}, \sigma_{b+1}, \omega)]$$

$$+ 2S(\sigma_b, \sigma_b, \omega) + S(k\omega, k\omega, \omega)]$$

$$= 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + \frac{q}{2}[S(\omega, \omega, k\omega) + S(\sigma_{b+1}, \sigma_{b+1}, \omega)]$$

$$+ 2S(\sigma_b, \sigma_b, \omega) + S(\omega, \omega, k\omega)]$$

$$(1-q)S(\omega, \omega, k\omega) \leq (2 + \frac{q}{2})S(\sigma_{b+1}, \sigma_{b+1}, \omega) + qS(\sigma_b, \sigma_b, \omega)$$

same as in case(i)

$$S(\omega, \omega, k\omega) \leq (\frac{4+3q}{2(1-q)})\alpha_b$$

In the last inequality of each case, the infimum on the right hand side is 0. So

$$S(\omega, \omega, k\omega) = 0$$

that is

$$k\omega = \omega$$

Thus k has a fixed point ω . Now we prove ω is unique If k has another fixed point ω_1 , then

$$k\omega_1 = \omega_1.$$

Now

$$S(\omega, \omega, \omega_1) = S(k\omega, k\omega, k\omega_1) \leq qU(\omega, \omega, \omega_1)$$

where

$$\begin{aligned}
U(\omega, \omega, \omega_1) &\in \left\{ \frac{1}{2} [S(\omega, \omega, k\omega) + S(\omega, \omega, k\omega) + S(\omega_1, \omega_1, k\omega_1)], \right. \\
&\quad \frac{1}{2} [S(\omega, \omega, k\omega) + S(\omega, \omega, k\omega_1) + S(\omega_1, \omega_1, k\omega)], \\
&\quad \left. \frac{1}{2} [S(\omega, \omega, k\omega_1) + S(\omega, \omega, k\omega) + S(\omega_1, \omega_1, k\omega)] \right\} \\
&= \{0, S(\omega, \omega, \omega_1), S(\omega, \omega, \omega_1)\}
\end{aligned}$$

This implies

$$S(\omega, \omega, \omega_1) = 0.$$

So $\omega = \omega_1$. Hence k has a unique fixed point.

Theorem 2.4 Let (\mathfrak{R}, S, V) be a vector S -metric space which is complete and V -Archimedean. Let $k: \mathfrak{R} \rightarrow \mathfrak{R}$ be a map such that

$$\begin{aligned}
S(k\hbar, k\vartheta, k\mu) &\leq \beta \max \{S(\hbar, \vartheta, \mu), S(\hbar, \hbar, k\hbar), S(\vartheta, \vartheta, k\vartheta), S(\mu, \mu, k\mu)\} + \\
&\quad \gamma \{S(\hbar, \hbar, k\vartheta) + S(\vartheta, \vartheta, k\mu) + S(\mu, \mu, k\hbar)\} + \delta \{S(\hbar, \hbar, k\mu) \\
&\quad + S(\vartheta, \vartheta, k\hbar) + S(\mu, \mu, k\vartheta)\}
\end{aligned}$$

where $\hbar, \vartheta, \mu \in \mathfrak{R}$, $\beta, \gamma, \delta > 0$ and $\beta + 4\gamma + 4\delta < 1$. Then k has a unique fixed point.

Proof. Assume that $\sigma_0 \in \mathfrak{R}$ and

$$\sigma_b = k\sigma_{b-1} \geq 1.$$

Then

$$\begin{aligned}
S(\sigma_b, \sigma_b, \sigma_{b+1}) &= S(k\sigma_{b-1}, k\sigma_{b-1}, k\sigma_b) \\
&\leq \beta \max \{S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b), S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_{b-1}), S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_b), \\
&\quad S(\sigma_b, \sigma_b, k\sigma_b)\} + \gamma \{S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_{b-1}) + S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_b) + \\
&\quad S(\sigma_b, \sigma_b, k\sigma_{b-1})\} + \delta \{S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_b) + S(\sigma_{b-1}, \sigma_{b-1}, k\sigma_{b-1}) \\
&\quad + S(\sigma_b, \sigma_b, k\sigma_{b-1})\} \\
\\
&= \beta \max \{S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b), S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b), S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b), \\
&\quad S(\sigma_b, \sigma_b, \sigma_{b+1})\} + \gamma \{S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_{b-1}, \sigma_{b-1}, \sigma_{b+1}) + \\
&\quad S(\sigma_b, \sigma_b, \sigma_b)\} + \delta \{S(\sigma_{b-1}, \sigma_{b-1}, \sigma_{b+1}) + S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) \\
&\quad + S(\sigma_b, \sigma_b, \sigma_b)\} \\
\\
&= \beta \max \{S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b), S(\sigma_b, \sigma_b, \sigma_{b+1})\} + (\gamma + \delta)
\end{aligned}$$

$$\{S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_{b-1}, \sigma_{b-1}, \sigma_{b+1})\}$$

$$\leq \beta \max\{S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b), S(\sigma_b, \sigma_b, \sigma_{b+1})\} + (\gamma + \delta)\{S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + 2S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_{b+1}, \sigma_{b+1}, \sigma_b)\}$$

$$S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \beta \max\{S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b), S(\sigma_b, \sigma_b, \sigma_{b+1})\} + (\gamma + \delta)\{3S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_b, \sigma_b, \sigma_{b+1})\} \quad (7)$$

Two cases arise:

$$(i) \max\{S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b), S(\sigma_b, \sigma_b, \sigma_{b+1})\} = S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$$

Then (7) becomes

$$S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \beta S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + (\gamma + \delta)\{3S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_b, \sigma_b, \sigma_{b+1})\}$$

$$(1 - \gamma - \delta)S(\sigma_b, \sigma_b, \sigma_{b+1}) = (\beta + 3\gamma + 3\delta)S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$$

$$S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \frac{\beta + 3\gamma + 3\delta}{1 - \gamma - \delta} S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$$

where

$$\frac{\beta + 3\gamma + 3\delta}{1 - \gamma - \delta} < 1.$$

$$(ii) \max\{S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b), S(\sigma_b, \sigma_b, \sigma_{b+1})\} = S(\sigma_b, \sigma_b, \sigma_{b+1})$$

Then (7) becomes

$$S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \beta S(\sigma_b, \sigma_b, \sigma_{b+1}) + (\gamma + \delta)\{3S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b) + S(\sigma_b, \sigma_b, \sigma_{b+1})\}$$

$$(1 - \beta - \gamma - \delta)S(\sigma_b, \sigma_b, \sigma_{b+1}) = (3\gamma + 3\delta)S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$$

$$S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \frac{3\gamma+3\delta}{1-\beta-\gamma-\delta} S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$$

where

$$\frac{3\gamma+3\delta}{1-\beta-\gamma-\delta} < 1.$$

combining both cases, we get

$$S(\sigma_b, \sigma_b, \sigma_{b+1}) \leq \lambda S(\sigma_{b-1}, \sigma_{b-1}, \sigma_b)$$

where

$$\lambda \in \left\{ \frac{\beta+3\gamma+3\delta}{1-\gamma-\delta}, \frac{3\gamma+3\delta}{1-\beta-\gamma-\delta} \right\} < 1.$$

Since V is Archimedean, by lemma (2.1) $\langle \sigma_b \rangle$ is a V -Cauchy sequence in \mathfrak{R} and \mathfrak{R} is V -complete, $\exists \omega \in \mathfrak{R}$ such that $\sigma_b \xrightarrow{S,V} \omega$. Hence there exist $\langle \alpha_b \rangle \in V$ such that $\alpha_b \downarrow 0$ and

$$S(\sigma_b, \sigma_b, \omega) \leq \alpha_b.$$

Now, we show that ω is a fixed point of k . Since

$$\begin{aligned} S(\omega, \omega, k\omega) &\leq 2S(\omega, \omega, k\sigma_b) + S(k\omega, k\omega, k\sigma_b) \\ &\leq 2S(\omega, \omega, \sigma_{b+1}) + \beta \max\{S(\omega, \omega, \sigma_b), S(\omega, \omega, k\omega), \\ &\quad S(\omega, \omega, k\omega), S(\sigma_b, \sigma_b, k\sigma_b)\} + \gamma\{S(\omega, \omega, k\omega) + \\ &\quad S(\omega, \omega, k\sigma_b) + S(\sigma_b, \sigma_b, k\omega)\} + \delta\{S(\omega, \omega, k\sigma_b) \\ &\quad + S(\omega, \omega, k\omega) + S(\sigma_b, \sigma_b, k\omega)\} \\ \\ &= 2S(\omega, \omega, \sigma_{b+1}) + \beta \max\{S(\omega, \omega, \sigma_b), S(\omega, \omega, k\omega), \\ &\quad S(\sigma_b, \sigma_b, k\sigma_b)\} + (\gamma + \delta)\{S(\omega, \omega, k\omega) + \\ &\quad S(\omega, \omega, k\sigma_b) + S(\sigma_b, \sigma_b, k\omega)\} \\ \\ &= 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + \beta \max\{S(\sigma_b, \sigma_b, \omega), S(\omega, \omega, k\omega), \\ &\quad S(\sigma_b, \sigma_b, \sigma_{b+1})\} + (\gamma + \delta)\{S(\omega, \omega, k\omega) + \\ &\quad S(\omega, \omega, \sigma_{b+1}) + S(\sigma_b, \sigma_b, k\omega)\} \\ \\ &\leq 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + \beta \max\{S(\sigma_b, \sigma_b, \omega), S(\omega, \omega, k\omega), \\ &\quad S(\sigma_b, \sigma_b, \sigma_{b+1})\} + (\gamma + \delta)\{S(\omega, \omega, k\omega) + S(\sigma_{b+1}, \sigma_{b+1}, \omega)\} \end{aligned}$$

$$+2S(\sigma_b, \sigma_b, \omega) + S(k\omega, k\omega, \omega)\}$$

$$\begin{aligned} &= 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + \beta \max\{S(\sigma_b, \sigma_b, \omega), S(\omega, \omega, k\omega), \\ &\quad S(\sigma_b, \sigma_b, \sigma_{b+1})\} + (\gamma + \delta)\{2S(\omega, \omega, k\omega) + S(\sigma_{b+1}, \sigma_{b+1}, \omega) \\ &\quad + 2S(\sigma_b, \sigma_b, \omega)\} \end{aligned}$$

$$\begin{aligned} (1 - 2\gamma - 2\delta)S(\omega, \omega, k\omega) &\leq 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + \beta \max\{S(\sigma_b, \sigma_b, \omega), S(\omega, \omega, k\omega), \\ &\quad S(\sigma_b, \sigma_b, \sigma_{b+1})\} + (\gamma + \delta)\{S(\sigma_{b+1}, \sigma_{b+1}, \omega) + 2S(\sigma_b, \sigma_b, \omega)\} \end{aligned}$$

Three cases arise:

$$(i) \max\{S(\sigma_b, \sigma_b, \omega), S(\omega, \omega, k\omega), S(\sigma_b, \sigma_b, \sigma_{b+1})\} = S(\sigma_b, \sigma_b, \omega)$$

$$\begin{aligned} (1 - 2\gamma - 2\delta)S(\omega, \omega, k\omega) &\leq 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + \beta S(\sigma_b, \sigma_b, \omega) + (\gamma + \delta) \\ &\quad \{S(\sigma_{b+1}, \sigma_{b+1}, \omega) + 2S(\sigma_b, \sigma_b, \omega)\} \end{aligned}$$

$$(1 - 2\gamma - 2\delta)S(\omega, \omega, k\omega) \leq (2 + \gamma + \delta)\alpha_{b+1} + (\beta + 2\gamma + 2\delta)\alpha_b$$

$$S(\omega, \omega, k\omega) \leq \frac{2 + \beta + 3\gamma + 3\delta}{1 - 2\gamma - 2\delta} \alpha_b$$

by using the condition $\beta + 4\gamma + 4\delta < 1$ we get $\frac{2 + \beta + 3\gamma + 3\delta}{1 - 2\gamma - 2\delta} > 0$.

So

$$S(\omega, \omega, k\omega) \leq \frac{2 + \beta + 3\gamma + 3\delta}{1 - 2\gamma - 2\delta} \alpha_b \downarrow 0.$$

This implies $S(\omega, \omega, k\omega) = 0$. So $k\omega = \omega$. Hence k has a fixed point ω .

$$(ii) \max\{S(\sigma_b, \sigma_b, \omega), S(\omega, \omega, k\omega), S(\sigma_b, \sigma_b, \sigma_{b+1})\} = S(\omega, \omega, k\omega)$$

$$\begin{aligned} (1 - 2\gamma - 2\delta)S(\omega, \omega, k\omega) &\leq 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + \beta S(\omega, \omega, k\omega) + (\gamma + \delta) \\ &\quad \{S(\sigma_{b+1}, \sigma_{b+1}, \omega) + 2S(\sigma_b, \sigma_b, \omega)\} \end{aligned}$$

$$\begin{aligned}
 (1 - \beta - 2\gamma - 2\delta)S(\omega, \omega, k\omega) &\leq (2 + \gamma + \delta)S(\sigma_{b+1}, \sigma_{b+1}, \omega) + (2\gamma + 2\delta)S(\sigma_b, \sigma_b, \omega) \\
 &\leq (2 + \gamma + \delta)\alpha_{b+1} + (2\gamma + 2\delta)\alpha_b \\
 S(\omega, \omega, k\omega) &\leq \frac{2+3\gamma+3\delta}{(1-\beta-2\gamma-2\delta)}\alpha_b
 \end{aligned}$$

by using the condition $\beta + 4\gamma + 4\delta < 1$ we get $\frac{2+3\gamma+3\delta}{(1-\beta-2\gamma-2\delta)} > 0$.

So

$$S(\omega, \omega, k\omega) \leq \frac{2+3\gamma+3\delta}{(1-\beta-2\gamma-2\delta)}\alpha_b \downarrow 0.$$

This implies $S(\omega, \omega, k\omega) = 0$. So $k\omega = \omega$. Hence k has a fixed point ω .

$$(iii) \max\{S(\sigma_b, \sigma_b, \omega), S(\omega, \omega, k\omega), S(\sigma_b, \sigma_b, \sigma_{b+1})\} = S(\sigma_b, \sigma_b, \sigma_{b+1})$$

$$\begin{aligned}
 (1 - 2\gamma - 2\delta)S(\omega, \omega, k\omega) &\leq 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + \beta S(\sigma_b, \sigma_b, \sigma_{b+1}) + (\gamma + \delta) \\
 &\quad \{S(\sigma_{b+1}, \sigma_{b+1}, \omega) + 2S(\sigma_b, \sigma_b, \omega)\}
 \end{aligned}$$

$$\begin{aligned}
 (1 - 2\gamma - 2\delta)S(\omega, \omega, k\omega) &\leq 2S(\sigma_{b+1}, \sigma_{b+1}, \omega) + \beta[2S(\sigma_b, \sigma_b, \omega) + S(\sigma_{b+1}, \sigma_{b+1}, \omega)] \\
 &\quad + (\gamma + \delta)\{S(\sigma_{b+1}, \sigma_{b+1}, \omega) + 2S(\sigma_b, \sigma_b, \omega)\} \\
 &\leq (2 + \beta + \gamma + \delta)S(\sigma_{b+1}, \sigma_{b+1}, \omega) + (\beta + 2\gamma + 2\delta) \\
 &\quad S(\sigma_b, \sigma_b, \omega) \\
 &\leq (2 + \beta + \gamma + \delta)\alpha_{b+1} + (\beta + 2\gamma + 2\delta)\alpha_b \\
 S(\omega, \omega, k\omega) &\leq \frac{2+2\beta+3\gamma+3\delta}{(1-2\gamma-2\delta)}\alpha_b
 \end{aligned}$$

by using the condition $\beta + 4\gamma + 4\delta < 1$ we get $\frac{2+2\beta+3\gamma+3\delta}{(1-2\gamma-2\delta)} > 0$.

So

$$S(\omega, \omega, k\omega) \leq \frac{2+2\beta+3\gamma+3\delta}{(1-2\gamma-2\delta)}\alpha_b \downarrow 0.$$

This implies $S(\omega, \omega, k\omega) = 0$. So $k\omega = \omega$. Hence k has a fixed point ω .

Now, we prove that ω is unique If k has another fixed point ω_1 , then

$$k\omega_1 = \omega_1.$$

Now

$$\begin{aligned} S(\omega, \omega, \omega_1) &= S(k\omega, k\omega, k\omega_1) \\ &\leq \beta \max \{S(\omega, \omega, \omega_1), S(\omega, \omega, k\omega), S(\omega, \omega, k\omega), S(\omega_1, \omega_1, k\omega_1)\} + \\ &\quad \gamma \{S(\omega, \omega, k\omega) + S(\omega, \omega, k\omega_1) + S(\omega_1, \omega_1, k\omega)\} + \delta \{S(\omega, \omega, k\omega_1) \\ &\quad + S(\omega, \omega, k\omega) + S(\omega_1, \omega_1, k\omega)\} \\ &\leq (\beta + 2\gamma + 2\delta)S(\omega, \omega, \omega_1) \end{aligned}$$

Since $\beta + 2\gamma + 2\delta < 1$, then we have $S(\omega, \omega, \omega_1) = 0$. So $\omega = \omega_1$. Hence k has a unique fixed point ω .

Corollary 2.5 Let (\mathfrak{R}, S, V) be a vector S -metric space which is complete and V -Archimedean. Suppose the mappings $k: Y \rightarrow Y$ satisfies

$$S(k\hbar, k\vartheta, k\eta) \leq qS(\hbar, \vartheta, \eta) \quad \forall \hbar, \vartheta, \eta \in \mathfrak{R}$$

where $q \in [0, 1)$. Then $k \in \mathfrak{R}$ has fixed point which is unique and for any $\eta_0 \in \mathfrak{R}$, iterative sequence $\{\eta_m\}$ defined by $\eta_m = k\eta_{m-1}$, for all $m \in \mathbb{N}$, V -converges to fixed point of k .

This is Banach Contraction Principle for vector valued S -metric space.

Example 2.6 Let $\mathfrak{R} = \mathbb{R}, V = \mathbb{R}^2$ and

$$S: \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \rightarrow \mathbb{R}^2$$

defined by

$$S(\hbar, \vartheta, \eta) = (\alpha(\hbar - \vartheta)^2, \beta(\vartheta - \eta)^2, \gamma(\eta - \hbar)^2)$$

where $\hbar, \vartheta, \mu \in \mathfrak{R}$, $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma > 0$

Then (\mathfrak{R}, S, V) is a complete vector S -metric space. Define $k: \mathfrak{R} \rightarrow \mathfrak{R}$ as

$$k\hbar = \frac{\hbar}{2} + 5 \quad \forall \hbar \in \mathfrak{R}.$$

this implies $\hbar = 10$ is a fixed point of k . Then

$$\begin{aligned}
 S(k\hbar, k\vartheta, k\eta) &= S\left(\frac{\hbar}{2} + 5, \frac{\vartheta}{2} + 5, \frac{\eta}{2} + 5\right) \\
 &= \frac{1}{4}(\alpha(\hbar - \vartheta)^2, \beta(\vartheta - \eta)^2, \gamma(\eta - \hbar)^2) \\
 &= \frac{1}{4}S(\hbar, \vartheta, \eta) \\
 &\leq qS(\hbar, \vartheta, \eta)
 \end{aligned}$$

where $q \in [0,1)$ is constant. This shows that k has unique fixed point at $\hbar = 10$.

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