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SAWI TRANSFORM OF HILFER-PRABHAKAR DERIVATIVES AND ITS APPLICATIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

This paper focuses on the Sawi transform of the Hilfer-Prabhakar fractional derivative and its regularised form. We will determine the solution to Cauchy-type fractional differential equations with Hilfer-Prabhakar fractional derivatives utilizing Sawi and Fourier's transformations involving the three-parameter Mittag-Leffler function.

Keywords and Phrases: Mittag-Leffler function, Sawi transform, Fourier transform, Fractional integral, Fractional derivative.

2020 MSC : 26A33, 42B10, 44A35.

1. INTRODUCTION

Fractional calculus is a branch of mathematics that deals with fractional integrals and fractional derivatives of real or complex orders. In recent years fractional calculus got much attention from researchers for mathematical modeling and different fields of study in science and technology. In the literature, different kinds of fractional integrals and derivatives are involved such as Riemann Liouville fractional integral and derivative, Caputo derivative, Hilfer derivative, etc. [8, 11]. The Prabhakar integral [15] is the modification of the Riemann-Liouville integral by extending its kernel with the three-parameter Mittag-Leffler function. The Hilfer-Prabhakar derivative and its regularized version were first introduced in[7]. Many researchers used Hilfer-Prabhakar fractional derivatives in modeling and other fields due to their special properties, especially the combination of several integral transforms like Laplace, Sumudu, Elzaki, Shehu, and others [7, 14, 17, 4]. In 2019, Mahgoub *et al.* [18] found a new integral transform called as Sawi transform. Now we can see that Sawi transform is widely used by researchers in science and engineering to solve different kinds of problems for integral and differential equations [10, 2, 13, 9, 5, 16].

In this paper, we find the Sawi transform of the Prabhakar integral, Prabhakar derivatives, Hilfer-Prabhakar derivative and their regularization versions. Further, we applied these results to some Cauchy-type fractional differential equations involving the Hilfer-Prabhakar fractional derivative presented in terms of the Mittag-Leffler type function.

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Definition 1.1 *[11] The Riemann Liouville fractional integral of order $\alpha > 0$ of a function $\psi(t)$ is given by*

$$ {}_0\mathcal{J}_t^{\alpha}\psi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\psi(\tau)d\tau, t > 0. \tag{1}$$

Definition 1.2 *[11] The Riemann Liouville fractional derivative of order α of a function $\psi(t)$ is given by*

$$ {}_0\mathcal{D}_t^{\alpha}\psi(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}\psi(\tau)d\tau, n-1 < \alpha < n, n \in \mathbb{N}. \tag{2}$$

Definition 1.3 *[11] The Caputo fractional derivative of order α of a function $\psi(t)$ is given by*

$$ {}_0^C\mathcal{D}_t^{\alpha}\psi(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\psi^{(n)}(\tau)d\tau, n-1 < \alpha < n, n \in \mathbb{N}. \tag{3}$$

Definition 1.4 *[8] Let $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, the Hilfer fractional derivative of $\psi(t)$ is defined as*

$$ {}_0\mathcal{D}_t^{\alpha,\beta}\psi(t) = \left({}_0\mathcal{J}_t^{\beta(1-\alpha)}\frac{d}{dt}({}_0\mathcal{J}_t^{(1-\alpha)(1-\beta)}\psi(t))\right). \tag{4}$$

Definition 1.5 *[15] The three-parameter Mittag-Leffler function given by Prabhakar is*

$$ E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{(z)^k}{k!}, z, \alpha, \beta, \gamma \in \mathbb{C}, \alpha > 0. \tag{5}$$

A generalization of (5) is given by Garra et al. [7] as

$$ e_{\alpha,\beta,\omega}^{\gamma}(t) = t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega t^{\alpha}), \tag{6}$$

where $\omega \in \mathbb{C}$ and $t > 0$.

Definition 1.6 *[15] Let $\psi \in L^1[0,b]; 0 < t < b < \infty$; $\psi * e_{\alpha,\beta,\omega}^{\gamma} \in W^{m,1}[0,b], m = [\beta]$. The Prabhakar fractional integral of $\psi(t)$ is given by*

$$ \begin{aligned} \mathcal{J}_{\alpha,\beta,\omega,0^+}^{\gamma}\psi(t) &= \int_0^t (t-\tau)^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega(t-\tau)^{\alpha})\psi(\tau)d\tau \\ &= (\psi * e_{\alpha,\beta,\omega}^{\gamma})(t), \end{aligned} \tag{7}$$

where $\alpha, \beta, \gamma, \omega \in \mathbb{C} and Re(\alpha), Re(\beta) > 0$.

Definition 1.7 *[15] Let $\psi \in L^1[0,b]$,$0 < t < b < \infty$. The Prabhakar fractional derivative of $\psi(t)$ is given by*

$$ \mathcal{D}_{\alpha,\beta,\omega,0^+}^{\gamma}\psi(t) = \frac{d^m}{dt^m}\mathcal{J}_{\alpha,m-\beta,\omega,0^+}^{-\gamma}\psi(t), \tag{8}$$

where $\alpha, \beta, \gamma, \omega \in \mathbb{C}$ with $Re(\alpha), Re(\beta) > 0$.

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Definition 1.8 *[7] Let $\psi \in AC^1[0,b], 0 < t < b < \infty$ and $m = [\beta]$. The regularized Prabhakar fractional derivative of $\psi(t)$ is given by*

$$^{C}\mathcal{D}^{\gamma}_{\alpha,\beta,\omega,0^+}\psi(t) = \mathcal{J}^{-\gamma}_{\alpha,m-\beta,\omega,0^+}\frac{d^m}{dt^m}\psi(t), \qquad (9)$$

where $\alpha, \beta, \gamma, \omega \in \mathbb{C}$ with $Re(\alpha), Re(\beta) > 0$.

Definition 1.9 *[7, 12] Let $\psi \in L^1[0,b], 0 < \beta < 1, 0 \leq \nu \leq 1, 0 < b < t < \infty, \psi * e^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega}(.) \in AC^1[0,b]$. The Hilfer-Prabhakar fractional derivative of $\psi(t)$ is given by*

$$\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(t) = \left(\mathcal{J}^{-\gamma\nu}_{\alpha,\nu(1-\beta),\omega,0^+}\frac{d}{dt}(\mathcal{J}^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}\psi)\right)(t), \qquad (10)$$

where $\gamma, \omega \in \mathbb{R}$ and $Re(\alpha) > 0$.

Definition 1.10 *[12] Let $\psi \in L^1[0,b], 0 < \beta < 1, 0 \leq \nu \leq 1, 0 < b < t < \infty$. The regularized Hilfer-Prabhakar fractional derivative of $\psi(t)$ is given by*

$$C\mathcal{D}^{\gamma,\beta,\nu\psi(t)}_{\alpha,\omega,0^+} = \left(\mathcal{J}^{-\gamma\nu}_{\alpha,\nu(1-\beta),\omega,0^+}\mathcal{J}^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}\frac{d}{dt}\psi\right)(t) \quad = \mathcal{J}^{-\gamma}_{\alpha,1-\beta,\omega,0^+}\frac{d}{dt}\psi(t), \qquad (11)$$

where $\gamma, \omega \in \mathbb{R}$ and $Re(\alpha) > 0$.

Definition 1.11 *[6] The Fourier integral transform of a function $\psi(x)$ is given by*

$$F[\psi(x),k] = \psi^*(k) = \int_{-\infty}^{\infty}\psi(x)exp(ikx)dx, \qquad (12)$$

where $\psi(x)$ is a piecewise continuous function defined on $(-\infty,\infty)$ in each partial interval and absolutely integrable in $(-\infty,\infty)$.

Definition 1.12 *[18] Let $\mathcal{T}(s)$ be the Sawi integral transform to the function $\psi(t)$ and is defined by*

$$Sa[\psi(t),s] = \mathcal{T}(s) = \frac{1}{s^2}\int_0^{\infty}\psi(t) \quad exp\left(\frac{-t}{s}\right)dt = \frac{1}{s}\int_0^{\infty}exp(-t)\psi(st)dt, s \in (\lambda_1,\lambda_2), \qquad (13)$$

over the set of functions

$$\mathcal{A} = \left\{\psi(t) | \exists M, \lambda_1, \lambda_2 > 0, k > 0, |\psi(t)| \leq Me^{\left(\frac{t}{\lambda_j}\right)}, if t \in (-1)^j \times [0,\infty)\right\},$$

The integral transform (13) exists for all $\psi(t) > k$.

Sawi transform is a linear operator with many properties similar to other integral transforms.

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Proposition 1.1 *[3] If $\mathcal{F}(s)$ and $\mathcal{G}(s)$ are the Sawi transforms of the functions $\psi(t)$ and $\phi(t)$ respectively then the convolution of Sawi is defined by*

$$Sa[\psi(t) * \phi(t), s] = s^2\mathcal{F}(s)\mathcal{G}(s), \tag{14}$$

or equivalently

$$T^{-1}[s^2\mathcal{F}(s)\mathcal{G}(s), t] = (\psi(t) * \phi(t)), \tag{15}$$

where

$$\psi(t) * \phi(t) = \int_0^\infty \psi(\tau)\phi(t-\tau)d\tau. \tag{16}$$

Theorem 1.1 *[18] Suppose $\mathcal{T}(s)$ is the Sawi transform of $\psi(t)$, then the Sawi transform of m^{th} derivative $\psi^{(m)}(t)$ is given by*

$$Sa[\psi^{(m)}(t), s] = s^{-m}\mathcal{T}(s) - \sum_{k=0}^{m-1} s^{k-m-1}\psi^{(k)}(0), m \geq 0. \tag{17}$$

2. MAIN RESULTS

Lemma 2.1 *The Sawi transform of the Mittag-Leffler type function (6) is given as*

$$Sa\left[t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega t^\alpha), s\right] = s^{\beta-2}(1-\omega s^\alpha)^{-\gamma}, \omega \in \mathbb{C}, \tag{18}$$

where $0 < \alpha < 1$ and $Re(\alpha), Re(\beta), Re(\gamma) > 0$.

Proof. Applying Sawi transform (13) on (6), we have

$$Sa\left[t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega t^\alpha), s\right] = \frac{1}{s}\int_0^\infty e^{-t}(st)^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega(st)^\alpha)dt, \tag{19}$$

using (5) in (19), we have

$$\begin{aligned} Sa\left[t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega t^\alpha), s\right] &= \frac{1}{s}s^{\beta-1}\int_0^\infty e^{-t}t^{\beta-1} \times \sum_{k=0}^{\infty}\frac{(\gamma)_k(\omega(st)^\alpha)^k}{\Gamma(\alpha k+\beta)k!}dt \\ &= s^{\beta-2}\sum_{k=0}^{\infty}\frac{(\gamma)_k(\omega s^\alpha)^k}{\Gamma(\alpha k+\beta)k!}\int_0^\infty e^{-t}t^{\alpha k+\beta-1}dt. \end{aligned}$$

For $|\omega s^\alpha| < 1$, on simplification, we arrive at (18).

Lemma 2.2 *The Sawi transform of Prabhakar fractional integral is defined as*

$$Sa[\mathcal{J}_{\alpha,\beta,\omega,0^+}^{\gamma}\psi(t), s] = s^\beta(1-\omega s^\alpha)^{-\gamma}\mathcal{T}(s). \tag{20}$$

Proof. Applying Sawi transform (13) on (7), we have

$$\begin{aligned} Sa[\mathcal{J}_{\alpha,\beta,\omega,0^+}^{\gamma}\psi(t), s] &= Sa\left[\int_0^t (t-\tau)^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega(t-\tau)^\alpha)\psi(\tau)d\tau, s\right] \\ &= Sa\left[(\psi * e_{\alpha,\beta,\omega}^{\gamma})(t), s\right], \end{aligned} \tag{21}$$

using (14) in (21), we have

$$Sa[\mathcal{J}_{\alpha,\beta,\omega,0^+}^{\gamma}\psi(t), s] = s^2 Sa\left[t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\omega t^\alpha), s\right] \times Sa[\psi(t), s]. \tag{22}$$

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Now, using (18) in (22), we arrive at (20).

Theorem 2.1 *The Sawi transform of the Prabhakar fractional derivative is defined as*

$$Sa[\mathcal{D}^{\gamma}_{\alpha,\beta,\omega,0^+}\psi(t),s] = s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\mathcal{T}(s) - \sum_{k=0}^{m-1} s^{k-m-1}\mathcal{D}^{\gamma}_{\alpha,k-m+\beta,\omega,0^+}\psi(t)|_{t=0}. \tag{23}$$

Proof. Applying Sawi transform (13) on definition (8), we have

$$Sa[\mathcal{D}^{\gamma}_{\alpha,\beta,\omega,0^+}\psi(t),s] = Sa\left[\frac{d^m}{dt^m}g(t),s\right], where g(t) = \mathcal{J}^{-\gamma}_{\alpha,m-\beta,\omega,0^+}\psi(t), \tag{24}$$

on using (17), we can write (24) as

$$Sa\quad[\mathcal{D}^{\gamma}_{\alpha,\beta,\omega,0^+}\psi(t),s] = s^{-m}Sa[g(t),s] - \sum_{k=0}^{m-1} s^{k-m-1}g^{(k)}(0), g^{(k)}(0) = \frac{d^k}{dt^k}\mathcal{J}^{-\gamma}_{\alpha,m-\beta,\omega,0^+}\psi(0). \tag{25}$$

Now, using result (20) in (25), we get

$$Sa\quad[\mathcal{D}^{\gamma}_{\alpha,\beta,\omega,0^+}\psi(t),s] = s^{-m}s^{m-\beta}(1-\omega s^{\alpha})^{\gamma}Sa[\psi(t),s] - \sum_{k=0}^{m-1} s^{k-m-1}\frac{d^k}{dt^k}\mathcal{J}^{-\gamma}_{\alpha,m-\beta,\omega,0^+}\psi(t)|_{t=0}.$$

In view of (8), we arrive at (23).

Theorem 2.2 *The Sawi transform of regularized Prabhakar fractional derivative is defined by*

$$Sa[{}^{C}\mathcal{D}^{\gamma}_{\alpha,\beta,\omega,0^+}\psi(t),s] = s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\mathcal{T}(s) - \sum_{k=0}^{m-1} s^{k-\beta-1}(1-\omega s^{\alpha})^{\gamma}\psi^{(k)}(0^+). \tag{26}$$

Proof. Applying Sawi transform (13) on definition (9), we have

$$Sa[{}^{C}\mathcal{D}^{\gamma}_{\alpha,\beta,\omega,0^+}\psi(t),s] = Sa\left[\mathcal{J}^{-\gamma}_{\alpha,m-\beta,\omega,0^+}h(t),s\right], where h(t) = \frac{d^m}{dt^m}\psi(t), \tag{27}$$

using result (20) in (27), we get

$$Sa[{}^{C}\mathcal{D}^{\gamma}_{\alpha,\beta,\omega,0^+}\psi(t),s] = s^{m-\beta}(1-\omega s^{\alpha})^{\gamma}Sa[h(t),s]. \tag{28}$$

Now, using (17) in (28), we get

$$Sa[{}^{C}\mathcal{D}^{\gamma}_{\alpha,\beta,\omega,0^+}\psi(t),s] \quad = s^{m-\beta}(1-\omega s^{\alpha})^{\gamma}\left[s^{-m}Sa[\psi(t),s] - \sum_{k=0}^{m-1} s^{k-m-1}\psi^{(k)}(0)\right].$$

On simplification, we arrive at (26).

Theorem 2.3 *Sawi transform of the Hilfer-Prabhakar fractional derivative is defined by*

$$Sa[\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(t),s] = s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\mathcal{T}(s) - s^{\nu(1-\beta)-2}(1-\omega s^{\alpha})^{\gamma\nu}\mathcal{J}^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}\psi(t)|_{t=0^+}. \tag{29}$$

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Proof. Applying Sawi transform (13) on definition (10), we have

$$Sa[\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(t),s] = Sa\left[\mathcal{J}^{-\gamma\nu}_{\alpha,\nu(1-\beta),\omega,0^+}k(t),s\right], where\ \ k(t) = \frac{d}{dt}\mathcal{J}^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}\psi(t), \quad (30)$$

on using result (20) in (30), we get

$$Sa\quad[\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(t),s] = s^{\nu(1-\beta)}(1-\omega s^{\alpha})^{\gamma\nu}Sa[k(t),s], \quad (31)$$

in view of (17) and (20), we get

$$\begin{aligned} Sa\quad&[\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(t),s] = \\ &s^{\nu(1-\beta)}(1-\omega s^{\alpha})^{\gamma\nu}\left[s^{-1}Sa[\mathcal{J}^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}\psi(t),s] - s^{-2}\mathcal{J}^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}\psi(0^+)\right] \\ &= s^{\nu(1-\beta)}(1-\omega s^{\alpha})^{\gamma\nu} \\ &\times\left[s^{(1-\nu)(1-\beta)-1}(1-\omega s^{\alpha})^{\gamma(1-\nu)}Sa[\psi(t),s] - s^{-2}\mathcal{J}^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}\psi(0^+)\right]. \end{aligned}$$

On simplification, we arrive at (29).

Theorem 2.4 *The Sawi transform of regularized Hilfer-Prabhakar fractional derivative is defined by*

$$Sa[{}^{C}\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(t),s] = s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\mathcal{T}(s) - s^{-\beta-1}(1-\omega s^{\alpha})^{\gamma}f(0^+) \quad (32)$$

Proof. Applying Sawi transform (13) on definition (11), we have

$$Sa[{}^{C}\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(t),s] = Sa\left[\mathcal{J}^{-\gamma}_{\alpha,1-\beta,\omega,0^+}z(t),s\right], where\ \ z(t) = \frac{d}{dt}\psi(t), \quad (33)$$

in view of (20), we get

$$Sa[{}^{C}\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(t),s] \quad = s^{1-\beta}(1-\omega s^{\alpha})^{\gamma}Sa[z(t),s], \quad (34)$$

using result (17) in (33), we get

$$Sa[{}^{C}\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(t),s] \quad = s^{1-\beta}(1-\omega s^{\alpha})^{\gamma}[s^{-1}Sa[\psi(t),s] - s^{-2}f(0^+)].$$

On simplification, we arrive at (32).

3. APPLICATIONS

This section provides the applications of the Sawi transform on Hilfer-Prabhakar fractional derivatives to the solution of some Cauchy-type fractional differential equations

Theorem 3.1 *If the average fluid velocity p and the dispersion coefficient ρ are constants. The solution of the generalized Cauchy-type problem for the fractional advection-dispersion equation*

$$\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(x,t) = -p\mathcal{D}_x\psi(x,t) + \rho\Delta^{\frac{\lambda}{2}}\psi(x,t), \quad (35)$$

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subjected to

$$\mathcal{J}^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}\psi(x,0^+) = g(x), \alpha > 0 \tag{36}$$

$$\lim_{x\to\infty}\psi(x,t) = 0, t \geq 0,$$

is given by

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty} t^{\nu(1-\beta)+n\beta-1}e^{(-ikx)}g(k)(ipk - \rho|k|^{\lambda})^n E^{\gamma(1+n)-\gamma\nu}_{\alpha,\nu(1-\beta)+\beta(n+1)}(\omega t^{\alpha})dk, \tag{37}$$

where $0 < \beta < 1, 0 \leq \nu \leq 1; \gamma, \omega, x \in \mathbb{R}, t \in \mathbb{R}^+$ and $\Delta^{\frac{\lambda}{2}}$ is discussed in [1].

Proof. For $\psi(x,t)$, let $\overline{\psi}(x,s)$ be the Sawi transform w.r.t t and $\psi^*(k,t)$ be the Fourier transform w.r.t x. On applying the Fourier and Sawi transforms to (35) and then using (29) and (36), we have

$$s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\overline{\psi}^*(k,s) - s^{\nu(1-\beta)-2}(1-\omega s^{\alpha})^{\gamma\nu}\mathcal{J}^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}\psi^*(k,0) = ipk\overline{\psi}^*(k,s) - \rho|k|^{\lambda}\overline{\psi}^*(k,s)$$

$$s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\overline{\psi}^*(k,s) - s^{\nu(1-\beta)-2}(1-\omega s^{\alpha})^{\gamma\nu}g^*(k) = ipk\overline{\psi}^*(k,s) - \rho|k|^{\lambda}\overline{\psi}^*(k,s)$$

$$\overline{\psi}^*(k,s)[s^{-\beta}(1-\omega s^{\alpha})^{\gamma} + \rho|k|^{\lambda} - ipk] = s^{\nu(1-\beta)-2}(1-\omega s^{\alpha})^{\gamma\nu}g^*(k).$$

On simplification,

$$\overline{\psi}^*(k,s) = \frac{s^{\nu(1-\beta)-2}(1-\omega s^{\alpha})^{\gamma\nu}g^*(k)}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\left[1+\frac{\rho|k|^{\lambda}-ipk}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}}\right]}$$

$$\overline{\psi}^*(k,s) = s^{\nu(1-\beta)+\beta-2}(1-\omega s^{\alpha})^{\gamma\nu-\gamma}g^*(k)\sum_{n=0}^{\infty}\left[\frac{-\rho|k|^{\lambda}+ipk}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}}\right]^n$$

$$\overline{\psi}^*(k,s) = \sum_{n=0}^{\infty}(ipk-\rho|k|^{\lambda})^n s^{\nu(1-\beta)+\beta+\beta n-2}(1-\omega s^{\alpha})^{\gamma\nu-\gamma n-\gamma}g^*(k).$$

For $\frac{\rho|k|^{\lambda}-ipk}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}} < 1$, we can arrive at (37) by using the inverse form of integral transforms (18) and (12).

Theorem 3.2 *If the average fluid velocity* p *and the dispersion coefficient* ρ *are constants. The solution of the generalized Cauchy-type problem for the fractional advection-dispersion equation*

$$^{C}\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(x,t) = -p\mathcal{D}_x\psi(x,t) + \rho\Delta^{\frac{\lambda}{2}}\psi(x,t), \tag{38}$$

subjected to

$$\psi(x,0^+) = g(x), x \in \mathbb{R} \tag{39}$$

$$\lim_{|x|\to\infty}\psi(x,t) = 0, t \geq 0,$$

is given by

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty} t^{\beta n}e^{(-ikx)}g(k)(ipk-\rho|k|^{\lambda})E^{\gamma n}_{\alpha,\beta(n+1)}(\omega t^{\alpha})dk, \tag{40}$$

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where $0<\beta<1, 0\le\nu\le 1; \gamma,\omega,x\in\mathbb{R}, t\in\mathbb{R}^+$ and $\Delta^{\frac{\lambda}{2}}$ is discussed in [1].

Proof. For $\psi(x,t)$, let $\overline{\psi}(x,s)$ be the Sawi transform w.r.t t and $\psi^*(k,t)$ be the Fourier transform w.r.t x. On applying the Fourier and Sawi transforms to (38) and then using (32) and (39), we have

$$s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\overline{\psi}^*(k,s)-s^{-\beta-1}(1-\omega s^{\alpha})^{\gamma}\psi^*(k,0)=ipk\overline{\psi}^*(k,s)-\rho|k|^{\lambda}\overline{\psi}^*(k,s)$$
$$\overline{\psi}^*(k,s)[s^{-\beta}(1-\omega s^{\alpha})^{\gamma}+\rho|k|^{\lambda}-ipk]=s^{-\beta-1}(1-\omega s^{\alpha})^{\gamma}g^*(k).$$

On simplification,

$$\overline{\psi}^*(k,s)=\frac{s^{-\beta-1}(1-\omega s^{\alpha})^{\gamma}g^*(k)}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\left[1+\frac{\rho|k|^{\lambda}-ipk}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}}\right]}$$
$$\overline{\psi}^*(k,s)=s^{-1}g^*(k)\sum_{n=0}^{\infty}\left[\frac{ipk-\rho|k|^{\lambda}}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}}\right]^n$$
$$\overline{\psi}^*(k,s)=g^*(k)\sum_{n=0}^{\infty}(ipk-\rho|k|^{\lambda})^n s^{\beta n-1}(1-\omega s^{\alpha})^{-\gamma n}.$$

For $\left[\frac{\rho|k|^{\lambda}-ipk}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}}\right]<1$, we can arrive at (40) by using the inverse form of integral transforms (18) and (12).

Theorem 3.3 *Let* $0<\beta<1, 0\le\nu\le 1$ *and* $\gamma\ge 0$. *The solution of Cauchy type fractional differential equation*

$$ {}^{C}\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,-\omega,0^+}\psi(x,t)=-\lambda(1-x)\psi(x,t), |x|\le 1, \quad (41)$$
$$f(x,0)=1, \quad (42)$$

is given by

$$\psi(x,t)=\sum_{n=0}^{\infty}(-\lambda)^n(1-x)^n t^{\beta n}E^{\gamma n}_{\alpha,\beta n+1}(-\omega t^{\alpha}), (43)$$

where $\omega,\gamma,x\in\mathbb{R}; \alpha,t,\lambda>0$.

Proof. Let $\overline{\psi}(x,s)$ be the Sawi transform of $\psi(x,t)$ w.r.t t. On applying the Sawi transform to (41) and then using (32) and (42), we have

$$s^{-\beta}(1+\omega s^{\alpha})^{\gamma}\overline{\psi}(k,s)-s^{2-\beta}(1+\omega s^{\alpha})^{\gamma}\overline{\psi}(x,0)=\lambda(1-x)\overline{\psi}(x,s)$$
$$\overline{\psi}(k,s)[s^{-\beta}(1+\omega s^{\alpha})^{\gamma}+\lambda(1-x)]=s^{-\beta-1}(1+\omega s^{\alpha})^{\gamma}.$$

On simplification,

$$\overline{\psi}(k,s)=\frac{s^{-\beta-1}(1+\omega s^{\alpha})^{\gamma}}{s^{-\beta}(1+\omega s^{\alpha})^{\gamma}\left[1+\frac{\lambda(1-x)}{s^{-\beta}(1+\omega s^{\alpha})^{\gamma}}\right]}$$
$$\overline{\psi}(k,s)=s^{-1}\sum_{n=0}^{\infty}\left[\frac{-\lambda(1-x)}{s^{-\beta}(1+\omega s^{\alpha})^{\gamma}}\right]^n$$
$$\overline{\psi}(k,s)=\sum_{n=0}^{\infty}(-\lambda)^n(1-x)^n s^{\beta n-1}(1+\omega s^{\alpha})^{-\gamma n}.$$

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For $\frac{\lambda(1-x)}{s^{-\beta}(1+\omega s^{\alpha})^{\gamma}} < 1$, we can arrive at (43) by using the inverse form of transform (18).

Theorem 3.4 *Let* $y(t) \in L^1[0,\infty); 0 < \beta < 1, 0 \le \nu \le 1;\ \gamma, \delta \ge 0$. *The solution of Cauchy type fractional differential equation*

$$\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(t) = \lambda \mathcal{J}^{\delta}_{\alpha,\beta,\omega,0^+}\psi(t) + y(t), \tag{44}$$

subjected to

$$\left(\mathcal{J}^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}\psi(t)\right)|_{t=0} = M, \tag{45}$$

is given by

$$\psi(t) = \sum_{n=0}^{\infty} \lambda^n \mathcal{J}^{\gamma+n(\delta+\gamma)}_{\alpha,\beta(2n+1),\omega,0^+}y(t) + M\sum_{n=0}^{\infty} \lambda^n \quad t^{\beta(2n+1)+\nu(1-\beta)-1} \times E^{\delta n+\gamma n+\gamma-\gamma\nu}_{\alpha,\nu(1-\beta)+\beta(2n+1)}(\omega t^{\alpha}), \tag{46}$$

where $\omega, \gamma, \lambda \in \mathbb{R}; M, t, \alpha > 0$.

Proof. Let $\mathcal{T}(s)$ be the Sawi transform of $\psi(t)$, applying the Sawi transform on (44) and then using (29), (45) and (20), we have

$$Sa\left[\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}y(t), s\right] = Sa[\lambda \mathcal{J}^{\delta}_{\alpha,\beta,\omega,0^+}\psi(t) + y(t)]$$

$$s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\mathcal{T}(s) - s^{\nu(1-\beta)-2}(1-\omega s^{\alpha})^{\gamma\nu} \quad \mathcal{J}^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}\psi(t)|_{t=0} = \lambda Sa[(\psi * e^{\delta}_{\alpha,\beta,\omega})(t), s] + Sa[y(t), s]$$

$$s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\mathcal{T}(s) - s^{\nu(1-\beta)-2}(1-\omega s^{\alpha})^{\gamma\nu}M = \lambda s^{\beta}(1-\omega s^{\alpha})^{-\delta}\mathcal{T}(s) + Sa[y(t), s].$$

On simplification,

$$\begin{aligned}
\mathcal{T}(s) &= \frac{Sa[y(t),s] + s^{\nu(1-\beta)-2}(1-\omega s^{\alpha})^{\gamma\nu}M}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\left[1-\frac{\lambda s^{\beta}(1-\omega s^{\alpha})^{-\delta}}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}}\right]} \\
&= \frac{Sa[y(t),s] + s^{\nu(1-\beta)-2}(1-\omega s^{\alpha})^{\gamma\nu}M}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}} \sum_{n=0}^{\infty} \left[\frac{\lambda s^{\beta}(1-\omega s^{\alpha})^{-\delta}}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}}\right]^n \\
&= \frac{Sa[y(t),s] + s^{\nu(1-\beta)-2}(1-\omega s^{\alpha})^{\gamma\nu}M}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}} \sum_{n=0}^{\infty} \lambda^n s^{2\beta n}(1-\omega s^{\alpha})^{-\delta n-\gamma n} \\
&= \left(Sa[y(t), s] + s^{\nu(1-\beta)-2}(1-\omega s^{\alpha})^{\gamma\nu}M\right) \sum_{n=0}^{\infty} \lambda^n s^{2\beta n+\beta}(1-\omega s^{\alpha})^{-\delta n-\gamma n-\gamma} \\
&= Sa[y(t), s] \sum_{n=0}^{\infty} \lambda^n s^{2\beta n+\beta}(1-\omega s^{\alpha})^{-\delta n-\gamma n-\gamma} \\
&\quad + M\sum_{n=0}^{\infty} \lambda^n s^{2\beta n+\beta+\nu(1-\beta)-2}(1-\omega s^{\alpha})^{-\delta n-\gamma n-\gamma+\gamma\nu}.
\end{aligned}$$

For $\left[\frac{\lambda s^{\beta}(1-\omega s^{\alpha})^{-\delta}}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}}\right] < 1$, we can arrive at (46) by using the inverse form of transform (18).

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Theorem 3.5 *Let* $0 < \beta < 1, 0 \le \nu \le 1$ *and* $\gamma \ge 0$. *The solution of the generalized Cauchy-type problem for the fractional heat equation*

$$\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(x,t) = M\frac{\partial^2}{\partial x^2}\psi(x,t), \tag{47}$$

subjected to

$$\mathcal{J}^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}\psi(x,t)|_{t=0} = g(x) \tag{48}$$

$$\lim_{x\to\infty}\psi(x,t) = 0,$$

is given by

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty} e^{-ikx}g(k)(-Mk^2)^n t^{\beta(n+1)-\nu(\beta-1)-1}E^{\gamma(n+1-\nu)}_{\alpha,\beta(n+1)+\nu(1-\beta)}dk, \tag{49}$$

where $\omega,\gamma,x \in \mathbb{R}; M,t,\alpha > 0$.

Proof. For $\psi(x,t)$, let $\overline{\psi}(x,s)$ be the Sawi transform w.r.t t and $\psi^*(k,t)$ be the Fourier transform w.r.t x. On applying the Fourier and Sawi transforms to (47) and then using (29) and (48), we have

$$s^{-\beta}(1-\omega s^\alpha)^\gamma\overline{\psi}^*(k,s) - s^{\nu(1-\beta)-2}(1-\omega s^\alpha)^{\gamma\nu}\mathcal{J}^{-\gamma(1-\nu)}_{\alpha,(1-\nu)(1-\beta),\omega,0^+}f(x,0) = -Mk^2\overline{\psi}^*(k,s)$$

$$s^{-\beta}(1-\omega s^\alpha)^\gamma\overline{\psi}^*(k,s) - s^{\nu(1-\beta)-2}(1-\omega s^\alpha)^{\gamma\nu}g^*(k) = -Mk^2\overline{\psi}^*(k,s)$$

$$\overline{\psi}^*(k,s)[s^{-\beta}(1-\omega s^\alpha)^\gamma + Mk^2] = s^{\nu(1-\beta)-2}(1-\omega s^\alpha)^{\gamma\nu}g^*(k).$$

On simplification,

$$\overline{\psi}^*(k,s) = \frac{s^{\nu(1-\beta)-2}(1-\omega s^\alpha)^{\gamma\nu}g^*(k)}{s^{-\beta}(1-\omega s^\alpha)^\gamma + Mk^2}$$

$$\overline{\psi}^*(k,s) = \frac{s^{\nu(1-\beta)-2}(1-\omega s^\alpha)^{\gamma\nu}g^*(k)}{s^{-\beta}(1-\omega s^\alpha)^\gamma\left[1+\frac{Mk^2}{s^{-\beta}(1-\omega s^\alpha)^\gamma}\right]}$$

$$\overline{\psi}^*(k,s) = s^{\nu(1-\beta)-2}(1-\omega s^\alpha)^{\gamma\nu-\gamma}g^*(k)\sum_{n=0}^{\infty}\left[\frac{-Mk^2}{s^{-\beta}(1-\omega s^\alpha)^\gamma}\right]^n$$

$$\overline{\psi}^*(k,s) = s^{\nu(1-\beta)-2}(1-\omega s^\alpha)^{\gamma\nu-\gamma}g^*(k)\sum_{n=0}^{\infty}(-Mk^2)^n s^{\beta n}(1-\omega s^\alpha)^{-\gamma n}$$

$$\overline{\psi}^*(k,s) = g^*(k)\sum_{n=0}^{\infty}(-Mk^2)^n s^{\beta n+\nu(1-\beta)+\beta-2}(1-\omega s^\alpha)^{\gamma\nu-\gamma n-\gamma}.$$

For $\left(\frac{Mk^2}{s^{-\beta}(1-\omega s^\alpha)^\gamma}\right) < 1$, we can arrive at (49) by using the inverse form of integral transforms (18) and (12).

Theorem 3.6 *Let* $0 < \beta < 1, 0 \le \nu \le 1$ *and* $\gamma \ge 0$. *The solution of the generalized Cauchy-type problem for the fractional heat equation*

$${}^{C}\mathcal{D}^{\gamma,\beta,\nu}_{\alpha,\omega,0^+}\psi(x,t) = N\frac{\partial^2}{\partial x^2}\psi(x,t), \tag{50}$$

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subjected to

$$\psi(x,0) = g(x) \tag{51}$$
$$\lim_{x\to\infty} \psi(x,t) = 0,$$

is given by

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} e^{-ikx} E_{\alpha,\beta n+1}^{\gamma n}(\omega t^{\alpha})(-Nk^2)^n t^{\beta n} g(k)dk, \tag{52}$$

where $\omega, \gamma, x \in \mathbb{R}; N, t, \alpha > 0$.

Proof. For $\psi(x,t)$, let $\overline{\psi}(x,s)$ be the Sawi transform w.r.t t and $\psi^*(k,t)$ be the Fourier transform w.r.t x. On applying the Fourier and Sawi transforms to (50) and then using (32) and (51), we have

$$s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\overline{\psi}^{*}(k,s) - s^{-\beta-1}(1-\omega s^{\alpha})^{\gamma}\psi^{*}(k,0) = -Nk^2\overline{\psi}^{*}(k,s)$$
$$\overline{\psi}^{*}(k,s)[s^{-\beta}(1-\omega s^{\alpha})^{\gamma} + Nk^2] = s^{-\beta-1}(1-\omega s^{\alpha})^{\gamma} g^{*}(k).$$

On simplification,

$$\overline{\psi}^{*}(k,s) = \frac{s^{-\beta-1}(1-\omega s^{\alpha})^{\gamma} g^{*}(k)}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}\left[1+\frac{Nk^2}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}}\right]}$$
$$\overline{\psi}^{*}(k,s) = s^{-1}g^{*}(k)\sum_{n=0}^{\infty}\left[\frac{-Nk^2}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}}\right]^n$$
$$\overline{\psi}^{*}(k,s) = s^{-1}g^{*}(k)\sum_{n=0}^{\infty}(-Nk^2)^n s^{\beta n}(1-\omega s^{\alpha})^{-\gamma n} g^{*}(k)$$
$$\overline{\psi}^{*}(k,s) = \sum_{n=0}^{\infty}(-Nk^2)^n s^{\beta n-1}(1-\omega s^{\alpha})^{-\gamma n} g^{*}(k).$$

For $\frac{Nk^2}{s^{-\beta}(1-\omega s^{\alpha})^{\gamma}} < 1$, we can arrive at (52) by using the inverse form of integral transforms (18) and (12).

4. CONCLUSION

This article begins with the Sawi transform on Hilfer-Prabhakar and regularizations of Hilfer-Prabhakar fractional derivatives. We then demonstrate how the three parameters of the Mittag-Leffler function can be utilized in Sawi and Fourier transformations to solve a set of Cauchy-type fractional differential equations with Hilfer-Prabhakar fractional derivatives. This result demonstrates the effectiveness of this method for solving fractional differential equations.

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