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ON μ^n − FUZZY GRAPHS

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Abstract

To avoid some limitations of fuzzy graphs, we introduced an advanced type of fuzzy graph named "μ^n − fuzzy graphs (MFG)". And we defined cut - node, Strength of connectedness, μ^n −fuzzy subgraph, partial μ^n −fuzzy subgraph, spanning μ^n −fuzzy subgraph, μ^n −fuzzy distance, μ^n −fuzzy eccentricity, in μ^n −fuzzy graphs and various properties and theorems of μ^n −fuzzy graphs are discussed.

Index Terms: Fuzzy graph, μ^n − fuzzy graphs, μ^n − fuzzy distance.

1 INTRODUCTION

A network structure can be used to solve the combinatorial issues in a variety of fields including geometry, algebra, number theory, topology, optimization and computer science. When either the set of vertices or the set of edges, or both, is unclear, the model becomes a fuzzy graph. In this paper, the authors introduced an advanced type of fuzzy graph named μ^n − fuzzy graphs. Figure

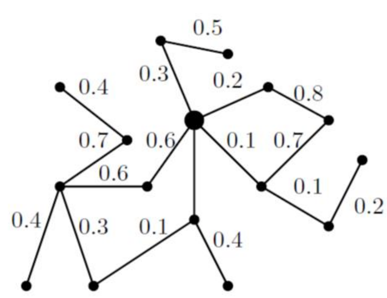

Figure 1 shows a real-life example of a fuzzy graph

People are represented as vertices in the example, and particular values have been assigned to them based on their friendship. If there is no relationship, we presume the value is zero, and if the friendship is extremely tight, the value will be near to one. i.e., depending on the degree of the friendship, all values will be between 0 and 1. But one of the drawbacks we experience with this graph is that we are not able to identify the traits based on which the value is given. In short, we are not able to analyse the process of evaluation. ie, a single real number is not sufficient to define

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a relation between two persons. This limitation can be overcome with the introduction of a new concept namely "$\mu^n -$ fuzzy graphs". In this model edge membership value is given as $n - tuple$ instead of a single real number. So that the above stated limitation of fuzzy graph is overcomes using this. Many authors, including Rosenfeld[3], Bhutani[1], Sunil Mathew[2], and Sunitha[4], established many connectivity notions in fuzzy graphs as a result of Zadeh[7][8], Yeh and Bang's[6] work. More related work can be seen in[5],[9]

A fuzzy graph ($f - graph$) is a triplet $G:(V,\sigma,\mu)$ where V vertex set, σ is a fuzzy subset of V and μ is a fuzzy relation on σ such that $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$ for all$u,v \in V$, where $\wedge$ represent the minimum. We assume that V is finite and non - empty, μ is reflexive and symmetric. In all the examples σ is chosen suitably. Also, we denote the underlying crisp graph by $G^*:(\sigma^*,\mu^*)$ where $\sigma^* = \{u \in S: \sigma(u) > 0\}$ and$\mu^* = \{(u,v) \in S \times S: \mu(u,v) > 0\}$. The fuzzy graph $\chi = (\tau,\nu)$ is called a fuzzy subgraph of $\xi = (\sigma,\mu)$ if $\tau(x) = \sigma(x)$ for all $x \in \tau^*$ and $\nu(x,y) = \mu(x,y)$ for all edges$(x,y) \in \nu^*, x,y \in V$. A fuzzy graph $H:(\tau,\nu)$ is called a partial fuzzy subgraph of $G:(\sigma,\mu)$ if $\tau(u) \leq \sigma(u)\ \forall u \in \tau^*$ and $\nu(u,v) \leq \mu(u,v) \forall (u,v) \in \nu^*$ and if in addition$\tau^* = \sigma^*$, then H is called a spanning fuzzy sub graph of$G$. The maximum of the strengths of all pathways between two nodes x and y is defined as the strength of connectedness between them and is denoted by$CONN_G(x,y)$. Through out, we assume that G is connected. A $x - y$ path P is called a strongest $x - y$ path if its strength equals$CONN_G(x,y)$. The eccentricity of a node u is defined and denoted by$e(u) = \max\{d(u,v)/v \in V\}$. The minimum and maximum eccentricities of all the nodes in a graph are known as the radius and diameter of the graph, respectively.

2 μ^n – FUZZY GRAPHS

We introduce a new model of fuzzy graph in this section.

Definition 2.1. A $\mu^n -$ fuzzy graph $K = (M,\sigma,\mu)$ is an algebraic structure of non-empty set M together with a pair of functions $\sigma: M \to [0,1]$ and $\mu: M \times M \to [0,1]^n$ such that for all $x,y \in M, \mu_i(x,y) \leq \sigma(x) \wedge \sigma(y)$ for $i = 1,2,\ldots,n$, where μ_i is the i th co - ordinate value or projection value of μ , ie, if $\mu(x,y) = (u_1,u_2,\ldots,u_n)$, $0 \leq u_i \leq 1$ for all $i = 1,2,\ldots,n$, then $\mu_i(x,y) = u_i$ for

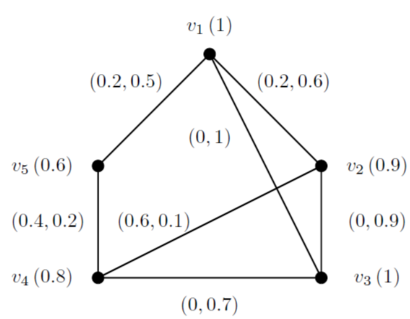

Figure 2

$i = 1,2,\ldots,n$ and μ is a symmetric fuzzy relation on σ and $\wedge$ represent the minimum. Here $\sigma(x)$ and $\mu(x,y)$ represent the membership values of the vertex x and of the edge (in n tuple) (x,y) in K respectively.

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For $n = 1$, it is simply a fuzzy graph, for $n = 2, \sigma : V \to [0,1]$ and $\mu: V \times V \to [0,1] \times [0,1]$ such that for all $x, y \in V$, $\mu_1(x,y) \leq \sigma(x) \wedge \sigma(y)$ and $\mu_2(x,y) \leq \sigma(x) \wedge \sigma(y)$. For example, consider the graph $G = (V, \sigma, \mu)$, $V = \{v_1, v_2, v_3, v_4, v_5\}$, the membership values of the vertices are $\sigma(v_1) = 1, \sigma(v_2) = 0.9, \sigma(v_3) = 1, \sigma(v_4) = 0.8, \sigma(v_5) = 0.6$ and the edge membership values are $\mu(v_1, v_2) = (0.2, 0.6)$, $\mu(v_2, v_3) = (0.0, 0.9)$, $\mu(v_3, v_4) = (0.0, 0.7)$, $\mu(v_4, v_5) = (0.4, 0.2)$, $\mu(v_5, v_1) = (0.2, 0.5)$, $\mu(v_1, v_3) = (0.0, 1.0)$, $\mu(v4, v2) = (0.6, 0.1)$.

Definition 2.2. The weight of an edge in a $\mu^n -$ fuzzy graph $K : (M, \sigma, \mu)$ is defined as the norm of the membership value of that edge. ie, weight of the edge (u,v), $w(u,v) = \|\mu(u,v)\|$, where $\|.\|$ is a norm on $\mathbb{R}^n$, we can choose suitable norms according to our requirements.

Definition 2.3. Given a $\mu^n -$ fuzzy graph $K : (M, \sigma, \mu)$, the underlying crisp graph is denoted as $K^* : (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and$\mu^* = \{(u,v) \in M \times M : w(u,v) > 0\}$.

Definition 2.4. A path in K is a sequence of vertices$x_0, x_1, \ldots, x_m$, such that $w\,(x_{i-1}, x_i) > 0$ for$i = 1, 2, \ldots., m$, the path is said to have length$m$. Two nodes that are joined by a path are said to be connected. A component of a $\mu^n -$ fuzzy graph is the $\mu^n -$ fuzzy sub graph such that any two vertices are connected by path. So, a $\mu^n -$ fuzzy graph is said to be connected if it has one component and disconnected otherwise.

Definition 2.5. Let $K : (M, \sigma, \mu)$ be a $\mu^n -$ fuzzy graph, then the $\mu^n -$ strength of the path $P: u_1 - u_2 - \cdots - u_n = e_1 - e_2 - \cdots - e_{n-1}$ is defined by $S_\mu(P) = \sum_{i=1}^{n-1} \|\mu(u_i, u_{i+1})\|$ OR $S_\mu(P) = \sum_{i=1}^{n-1} \|\mu(e_i)\|$, where $u_1, u_2, \ldots, u_n$ are the vertices of the path P and $e_1, e_2, \ldots, e_{n-1}$ are the corresponding edges and $\|.\|$ is a norm on $\mathbb{R}^n$, we can choose suitable norms according to our requirements.

3. $\mu^n -$ FUZZY DISTANCE

Definition 3.1. Let $K : (M, \sigma, \mu)$ be a $\mu^n -$ fuzzy graph, then the $\mu^n -$ fuzzy distance between two nodes u and v in G is defined to be $d_\mu(u,v) = \wedge_P \{S_\mu(P)/P$ is a $u - v$ path$\}$, where $S_\mu(P)$ is the $\mu^n -$ strength of the path P and $\wedge$ represents the minimum.

Remark 3.1. For a path $P: u_1 - u_2 - \cdots - u_n$ in a $\mu^n -$ fuzzy graph $: (M, \sigma, \mu)$, $\|\mu(u_i, u_{i+1})\| = 0$ for some i iff$u_1 = u_n$.

Theorem 3.1. d_μ is a metric on $\mu^n -$ fuzzy graph $K : (M, \sigma, \mu)$.

(i) $d_e(u,v) \geq 0$

Clearly $\|\mu(u_i, u_{i+1})\| \geq 0$ as $\|\cdot\|$ is a norm on $\mathbb{R}^n \Rightarrow \sum_{i=1}^{n-1} \|\mu(u_i, u_{i+1})\| \geq 0 \quad \Rightarrow S_\mu(P) \geq 0$ for each $u - v$ path $P \Rightarrow \wedge_P \{S_\mu(P)\} \geq 0 \Rightarrow d_\mu(u,v) \geq 0$

(ii) $d_\mu(u,v) = 0 \Leftrightarrow u = v$

$d_\mu(u,v) = 0 \Leftrightarrow \wedge_P \{S_\mu(P) : P$ is a $u - v$ path$\} = 0 \Leftrightarrow S_\mu(P) = 0$ for a $u - v$ path $P \Leftrightarrow \sum_{i=1}^{n-1} \|\mu(u_i, u_{i+1})\| = 0 \Leftrightarrow \|\mu(u_i, u_{i+1})\| = 0$ for $i = 1,2, \ldots, n-1 \quad \Leftrightarrow \mu(u_i, u_{i+1}) = 0$ for $i =$

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$1,2,\ldots,n-1$. For a path P connecting u and v , $\mu(u_i,u_{i+1})=0$ for $i=1,2,\ldots,n-1 \Leftrightarrow u=u_1=u_2=\cdots=u_n=v \Leftrightarrow u=v$. So, $d_\mu(u,v)=0 \Leftrightarrow u=v$.

(iii) $d_\mu(u,v)=d_\mu(v,u)$

$d_\mu(u,v) = \wedge_P \{S_\mu(P)/P$ is a $u-v$ path, $S(P)$ is the strength of the $P\} = \wedge_P \{S_\mu(P)/P$ is a $v-u$ path, $S_\mu(P)$ is the strength of the $P\} = d_\mu(v,u)$. ie, $d_\mu(u,v)=d_\mu(v,u)$.

(iv) $d_\mu(u,w) \le d_\mu(u,v) + d_\mu(v,w)$

Now for triangle inequality, let P_1 be a $u-v$ path such that $d(u,v)=L(P_1)$ and P_2 be a $v-w$ path such that$d(v,w)=L(P_2)$. The path P_1 followed by P_2 is a $u-w$ walk and since a path can be extracted from every walk, there is a $u-w$ path in G whose length is at most $d_\mu(u,v)+d_\mu(v,w)$. Therefore $d_\mu(u,w) \le d_\mu(u,v)+d_\mu(v,w)$.

So $d_\mu(u,w) \le d_\mu(u,v)+d_\mu(v,w)$.

Definition 3.2. The μ^n- fuzzy graph $\chi=(\tau,\nu)$ is called a μ^n- fuzzy subgraph of $\xi=(\sigma,\mu)$ if $\tau(x)=\sigma(x)$ for all $x\in\tau^*$ and $\nu_i(x,y)=\mu_i(x,y), i=1,2,\ldots,n$ for all edges $(x,y)\in\nu^*, x,y\in$

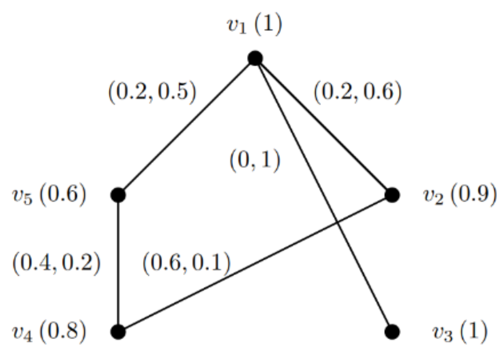

Figure 3

M

Here, figure 3 is a μ^2- fuzzy subgraph of figure 2.

Definition 3.3. A μ^n- fuzzy graph $H:(\tau,\nu)$ is called a partial μ^n- fuzzy subgraph of $G:(\sigma,\mu)$ if $\tau(u)\le\sigma(u)\forall u\in\tau^*$ and $\nu_i(u,v)\le\mu_i(u,v)\forall(u,v)\in\nu^*$ for $i=1,2,\ldots,n$ and if in addition $\tau^*=\sigma^*$, then H is called a spanning μ^n- fuzzy subgraph of G.

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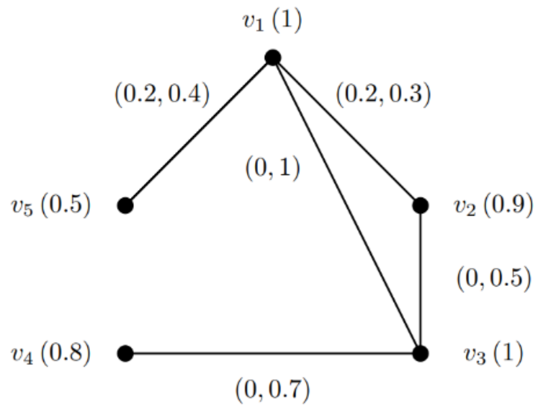

Figure 4

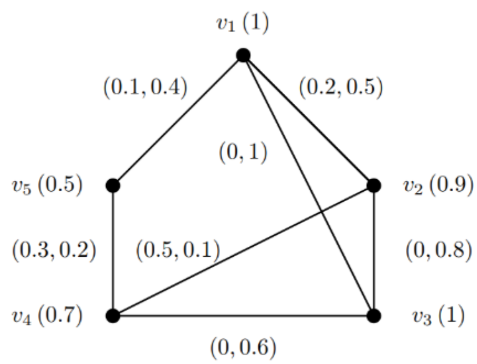

Figure 5

Here figure 4 is a partial μ^2- fuzzy subgraph of figure 2 and figure 5 is a spanning μ^2- fuzzy subgraph of figure 2.

Definition 3.4. The strength of connectedness between two nodes x and y is defined as the maximum of the μ^n- strengths of all paths between x and y and is denoted by $\mu^\infty(x,y)$ or $CONN_G(x,y)$. Through out, we assume that G is connected. An $x-y$ path P is called a strongest $x-y$ path if its strength equals $CONN_G(x,y)$.

Definition 3.5. Let $H-(x,y)$ be the μ^n- fuzzy graph obtained from H by replacing $\mu_i(x,y)$ by 0 for $i=1,2,\ldots,n$. We call the arc (x,y) strong in H if $\mu_i(x,y)>0$ for some $i=1,2,\ldots,n$ and$\|\mu(x,y)\| \geq CONN_{H-(x,y)}(x,y)$, otherwise the arc (x,y) is called weak.

Definition 3.6. Let $K:(M,\sigma,\mu)$ be a connected μ^n- fuzzy graph. Then the μ^n- fuzzy eccentricity of a node $u \in V(K)$ is defined and denoted by$e_\mu(u) = \vee_{v\in M}\{d_\mu(u,v)\}$, where $\vee$ represents the

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maximum. The set of all $\mu^n -$ fuzzy eccentric nodes of a node u is denoted byu^*. K is a unique eccentric node (u.e.n) $\mu^n -$ fuzzy graph if each node in K has a unique eccentric node.

Definition 3.7. The minimum of the $\mu^n -$ fuzzy eccentricities of all nodes is called the $\mu^n -$ radius of the graph$G$. It is denoted as$r_\mu(G)$. Thus$r_\mu(G) = \wedge_{u\in V} \{e_\mu(u)\}$.

Definition 3.8. The maximum of the $\mu^n -$ fuzzy eccentricities of all the nodes is called the $\mu^n -$ fuzzy diameter of the graph$G$. It is denoted as$d_\mu(G)$. That is, $d_\mu(G) = \vee_{u\in V} \{e_\mu(u)\}$.

Definition 3.9. A node $v \in V(K)$ is called a $\mu^n -$ fuzzy eccentric node of another node u if$e_\mu(u) = d_\mu(u,v)$.

Definition 3.10. A node u is a central node or $\mu^n -$fuzzy radial node if $e_\mu(u) = r_\mu(K)$(nodes with minimum $\mu^n -$ fuzzy eccentricity), and $C_\mu(K)$ is the set of all central nodes. The $\mu^n -$ fuzzy subgraph induced by $C_\mu(K)$ denoted by $\langle C_\mu(K)\rangle = H\colon(V,\tau,v)$ is called the $\mu^n -$ center of K. A connected $\mu^n -$ fuzzy graph K is self centered if each node is a $\mu^n -$ central node i.e. $K \approx H$. A node u is a peripheral node or $\mu^n -$ fuzzy diametral nodes if $e_\mu(u) = d_\mu(K)$(nodes with maximum $\mu^n -$ fuzzy eccentricity). ie, the subgraph induced by the set of all $\mu^n -$ fuzzy central nodes is called the $\mu^n -$ center of the $\mu^n -$ fuzzy graph K and the subgraph induced by the set of all $\mu^n -$ fuzzy diametral nodes is called the $\mu^n -$ periphery of the $\mu^n -$ fuzzy graph G.

Similar to the classical distance in graphs, we have the following result.

Theorem 3.2. In any connected $\mu^n -$ fuzzy graph$K\colon(M,\sigma,\mu)$, $r_\mu(K) \le d_\mu(K) \le 2r_\mu(K)$.

Proof. The first inequality follows from the definition itself. To prove the other, let u and v be any two nodes such that$d_\mu(u,v) = d_\mu(K)$. Let w be any $\mu^n -$ fuzzy central node of$K$. Then by triangle inequality, $d_\mu(u,v) \le d_\mu(u,w) + d_\mu(w,v)$. But $d_\mu(u,w) \le v(K), d_\mu(w,v) \le r_\mu(K)$ and $r_\mu(K) \le d_\mu(K)$. Thus$d_\mu(u,v) \le r_\mu(K) + r_\mu(K) = 2\,r_\mu(K) \Rightarrow d_\mu(K) \le 2\,r_\mu(K) \Rightarrow r_\mu(K) \le d_\mu(K) \le 2\,r_\mu(K)$.

Definition 3.11 Let w be any vertex and let K' be the partial $\mu^n -$ fuzzy subgraph of K obtained by deleting the vertex$w$. That is K' is the partial $\mu^n -$ fuzzy subgraph of K such that $\sigma'(w) = 0, \sigma = \sigma'$ for all other vertices, $\mu'(wz) = (0,0,\ldots,0)$ for all vertices z and $\mu' = \mu$ for all other edges. We call w a $\mu^n -$ fuzzy cut vertex in K if ${\mu'}^\infty(u,v) < \mu^\infty(u,v)$ for some u,v in Vsuch that$u \ne w \ne v$. In words, w is a $\mu^n -$ fuzzy cutvertex If and only if there exist u,v distinct from w such that w is on every strongest path from u to$v$.

CONCLUSION

In this paper, the author introduced a new type of fuzzy graph namely $\mu^n -$ fuzzy graph. The concept of $\mu^n -$ fuzzy distance, eccentricity, strength of connectedness between two nodes, … are also introduced.

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