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A GENERALIZATION OF TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS FROM n^{th} ROOT OF UNITY

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Abstract

This paper introduces generalized trigonometric and hyperbolic functions defined by exponential representations involving cube roots of -1. Through the illustration of their definitions, proofs, and properties, these functions are explained as solutions to certain non-homogeneous equations. A characterization theorem is presented, demonstrating uniqueness based on specified conditions. The relationship between roots of generalized trigonometric functions and hyperbolic functions are examined. Furthermore, the application of these functions in solving a differential equation is demonstrated.

Keywords: Trigonometric Functons, Hyperbolic Functions, nth Root of Unity.

2020 Mathematics Subject Classification: 11L03, 11Lxx, 12Dxx.

1. INTRODUCTION

Generalized trigonometric functions expand upon the traditional sine, cosine, and tangent functions, broadening their application beyond acute angles and unit circles. In references [2,3], these functions are redefined from various perspectives, exploring their properties in diverse contexts. Reference [4] adopts a unit circle methodology to construct sine and cosine functions along with their inverses. Also this work extends generalized trigonometric functions characterized by two parameters. It is well known from basic calculus is that $\sin x = \frac{e^{ix} - e^{ix}}{2i}$ and $\cos x = \frac{e^{ix} + e^{ix}}{2}$. Obviously this can be extended: Let $\omega = \frac{1+i\sqrt{3}}{2}$ be a cube root of -1. Then the powers of ω exhibit distinct patterns, distinguishing the sixth roots of unity into two categories: -1 and $-\omega^2$ as the cube roots of -1, and 1, $-\omega$ and ω^2 are the cube roots of unity.

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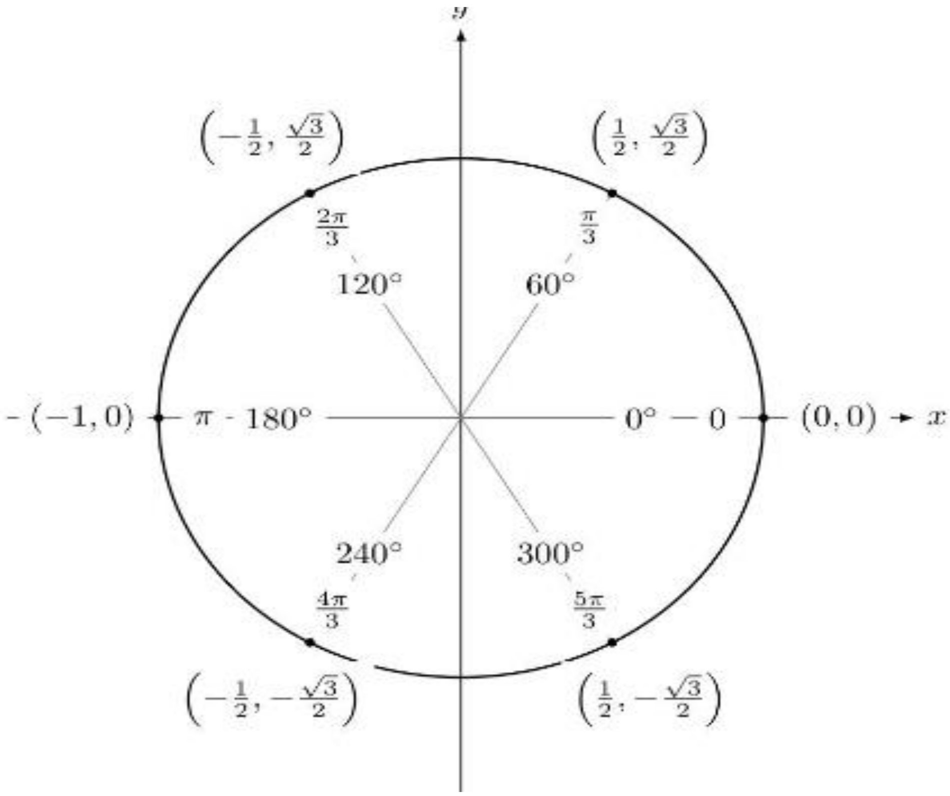

It is well known that.

$$e^{\omega x} = \sum_{k=0}^{\infty} \frac{(\omega x)^k}{k!}.$$

Since ω is a cube root of -1, $\omega^4 = -\omega,\ \omega^5 = -\omega^2,\ \omega^6 = 1$ and so on.

Thus $$e^{\omega x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{3k}}{(3k)!} + \omega \sum_{k=0}^{\infty} (-1)^k \frac{x^{3k+1}}{(3k+1)!} + \omega^2 \sum_{k=0}^{\infty} (-1)^k \frac{x^{3k+2}}{(3k+2)!}$$

2. GENERALIZED TRIGONOMETRIC FUNCTIONS

This section describes the generalized trigonometric functions and their properties.

Definition 1.1: For $n \geq 3$, the generalized trigonometric function

$$T_G(n,r)(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{nk+r}}{(nk+r)!},\ r = 0,1,2,\ldots,n-1.$$

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Theorem 1.2: Let ω be a cube root of -1. Then we have the following:

$$(i)T_G(3,0)(x)=\frac{e^{-x}+e^{\omega x}+e^{-\omega^2 x}}{3};$$

$$(ii)T_G(3,1)(x)=\frac{-e^{-x}-\omega^2 e^{\omega x}+\omega e^{-\omega^2 x}}{3};$$

$$(iii)T_G(3,2)(x)=\frac{e^{-x}-\omega e^{\omega x}+\omega^2 e^{-\omega^2 x}}{3}.$$

Proof: By the definition of the exponential function

$$e^{\omega x}=\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k}}{(3k)!}+\omega\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k+1}}{(3k+1)!}+\omega^2\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k+2}}{(3k+2)!}$$

$$=T_G(3,0)(x)+\omega T_G(3,1)(x)+\omega^2 T_G(3,2)(x)............(1)$$

$$e^{-\omega^2 x}=\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k}}{(3k)!}-\omega^2\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k+1}}{(3k+1)!}-\omega\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k+2}}{(3k+2)!}$$

$$=T_G(3,0)(x)-\omega^2 T_G(3,1)(x)-\omega T_G(3,2)(x)............(2)$$

$$e^{-x}=\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k}}{(3k)!}-\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k+1}}{(3k+1)!}+\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k+2}}{(3k+2)!}$$

$$=T_G(3,0)(x)-T_G(3,1)(x)+T_G(3,2)(x)\quad ....(3)$$

Solving the non-homogeneous equations (1), (2) and (3) in terms of $T_G(3,0)(x)$, $T_G(3,1)(x)$ and $T_G(3,2)(x)$ we get the required formula.

Properties of generalized trigonometric functions

The generalized trigonometric functions $T_G(3,0)(x)$, $T_G(3,1)(x)$ and $T_G(3,2)(x)$ have the following properties. This describes they are the solution of the differential equation $y'''+y=0$.

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Theorem 1.3:

$(i) T_G(3,0)(0) = 1;$

$$T_G(3,0)'(x) = -T_G(3,2)(x);$$

$$T_G(3,0)''(x) = -T_G(3,1)(x);$$

$$T_G(3,0)'''(x) = -T_G(3,0)(x).$$

$(ii) T_G(3,1)(0) = 0;$

$$T_G(3,1)'(x) = T_G(3,0)(x);$$

$$T_G(3,1)''(x) = -T_G(3,2)(x);$$

$$T_G(3,1)'''(x) = -T_G(3,1)(x).$$

$(iii) T_G(3,2)(0) = 0;$

$$T_G(3,2)'(x) = T_G(3,1)(x);$$

$$T_G(3,2)''(x) = T_G(3,0)(x);$$

$$T_G(3,2)'''(x) = -T_G(3,2)(x).$$

Proof: Immediately follows from the theorem 1.2.

Theorem 1.4: Characterization Theorem:

Let $f(x), g(x)$ and $h(x)$ are complex valued functions such that

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$(i)\ f(0)=1,\ f'(x)=-h(x),\ f''(x)=-g(x),\ f'''(x)=-f(x);$

$(ii)\ g(0)=0,\ g'(x)=f(x),\ g''(x)=-h(x),\ g'''(x)=-g(x);$

$(iii)\ h(0)=0,\ h'(x)=g(x),\ h''(x)=f(x),\ h'''(x)=-h(x),$

Then $f(x)=T_G(3,0)(x),\ \mathrm{g}(x)=T_G(3,1)(x)$ and $\mathrm{h}(x)=T_G(3,2)(x).$

Proof: Since $f(x), g(x)$ and $h(x)$ satisfies conditions in (i),(ii) and (iii), it is enough to prove that the functions are uniquely determined by (i),(ii) and (iii).

4. GENERALIZED HYPERBOLIC FUNCTION

Definition 1.5: For $n \geq 3$, the generalized hyperbolic function

$$H_G(n,r)(x)=\sum_{k=0}^{\infty}\frac{x^{nk+r}}{(nk+r)!},\ r=0,1,2,\dots,n-1.$$

Theorem1.6: Let ω be a cube root of -1. Then we have the following:

$$(i) H_G(3,0)(x)=\frac{e^{-x}+e^{-\omega x}+e^{\omega^2 x}}{3};$$

$$(ii) H_G(3,1)(x)=\frac{e^{-x}+\omega^2 e^{\omega x}-\omega e^{-\omega^2 x}}{3};$$

$$(iii) H_G(3,2)(x)=\frac{e^{x}-\omega e^{-\omega x}+\omega^2 e^{-\omega^2 x}}{3}.$$

Proof: By the definition of the exponential function

$$e^{-\omega x}=\sum_{k=0}^{\infty}\frac{x^{3k}}{(3k)!}-\omega\sum_{k=0}^{\infty}\frac{x^{3k+1}}{(3k+1)!}+\omega^2\sum_{k=0}^{\infty}\frac{x^{3k+2}}{(3k+2)!}$$

$$=\mathrm{H}_G(3,0)(x)-\omega H_G(3,1)(x)+\omega^2 H_G(3,2)(x)\dots\dots\dots\dots(8)$$

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$$e^{\omega^2 x} = \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!} + \omega^2 \sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)!} - \omega \sum_{k=0}^{\infty} \frac{x^{3k+2}}{(3k+2)!}$$

$$= \mathrm{H}_G(3,0)(x) + \omega^2 T_G(3,1)(x) - \omega H_G(3,2)(x) \ldots\ldots\ldots\ldots(9)$$

$$e^{x} = \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!} + \sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)!} + \sum_{k=0}^{\infty} \frac{x^{3k+2}}{(3k+2)!}$$

$$= \mathrm{H}_G(3,0)(x) + H_G(3,1)(x) + H_G(3,2)(x) \quad \ldots\ldots\ldots\ldots(10)$$

Solving the non-homogeneous equations (8),(9) and (10) in terms of $H_G(3,0)(x)$, $H_G(3,1)(x)$ and $H_G(3,2)(x)$, we get the required formula.

Relation between roots of generalized trigonometric functions and generalized hyperbolic functions.

$(i) T_G(3,0)(-x) = H_G(3,0)(x),\ T_G(3,1)(-x) = -H_G(3,1)(x),\ T_G(3,2)(-x) = H_G(3,2)(x);$

$(ii) T_G(3,0)(\omega x) = H_G(3,0)(x),\ T_G(3,1)(\omega x) = \omega H_G(3,1)(x),\ T_G(3,2)(\omega x) = \omega^2 H_G(3,2)(x);$

$(iii) T_G(3,0)(-\omega x) = T_G(3,0)(x),\ T_G(3,1)(-\omega x) = -\omega T_G(3,1)(x),\ T_G(3,2)(-\omega x) = \omega^2 T_G(3,2)(x);$

$(iv) T_G(3,0)(\omega^2 x) = T_G(3,0)(x),\ T_G(3,1)(\omega^2 x) = \omega^2 T_G(3,1)(x),\ T_G(3,2)(\omega^2 x) = -\omega T_G(3,2)(x);$

$(v) T_G(3,0)(-\omega^2 x) = H_G(3,0)(x),\ T_G(3,1)(-\omega^2 x) = -\omega^2 H_G(3,1)(x),\ T_G(3,2)(\omega^2 x) = -\omega H_G(3,2)(x).$

Applications

Solve the differential equation $y''' + y = 0$ using generalized trigonometric functions.

Solution:

We have from theorem 1.3 , the three generalized trigonometric functions satisfy the given differential equation and hence any linear combination of these functions will also be a solution of the differential equation. In particular $e^{-x}, e^{-\omega x}$ and $e^{-\omega^2 x}$ are all solutions and since the wronskian $W\left(e^{-x}, e^{-\omega x}, e^{-\omega^2 x}\right) \neq 0$, $\mathrm{y} = \mathrm{c}_1 e^{-x} + c_2 e^{-\omega x} + \mathrm{c}_3\, e^{-\omega^2 x}$ is the general solution of the given equation.

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5. CONCLUSION

The generalized trigonometric functions and their properties explains a rich mathematical landscape beyond traditional trigonometry. Through Definitions 1.1 and 1.5 and the subsequent theorems and proofs, we establish foundational understanding and relationships within these functions. Theorems 1.2 and 1.6 demonstrate key characteristics of these functions, while Theorem 1.3 establishes their relevance in differential equations. The characterization theorem (Theorem 1.4) further defines their uniqueness based on specific conditions. Moreover, the practical application shows in solving a differential equation underscores the utility of generalized trigonometric functions in real-world problem-solving.

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