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STUDY ABOUT GENERALISATION OF FIXED POINT THEOREM ON PARTIAL METRIC SPACE

ANSAM GHAZI NSAIF ALBU_AMER

Lecturer, Faculty of Basic Education,University of Wassit, Iraq
Emailansaif@uowasit.edu.iq

Abstract

FXP (Fixed point) theory is remarkably important in the world of mathematics because it has numerous applications in a variety of fields. Many authors demonstrated that M-ss results can be utilised to derive FXP generalisations in partial **M-ss (metric space)**. The subject of partial metrics is introduced, as well as the Caristi theorem in partial M-s, in this research paper. In quasi M-s, the Caristi theorem is also given and explained. Finally, some partial M-s generalisations of the Caristi transform are investigated.

Keywords: FXP Theorem; Partial M-s; Kirk's Caristi transform.

Introduction

Banach was the first one in 1922[1] to study full-scale FXP theory. Unlike the fixed-point theorems of Brouwer and Schauder, the Banach fixed-point theorem guarantees the existence of a transformation's FXP, as well as its uniqueness and how to find it. "FXP theory" studies are used in modern mathematics to demonstrate the existence and uniqueness of solutions to equations such as differential and integral equations. In addition, this theory has found some application areas in different disciplines such as physics, engineering, medicine, communication and economics apart from mathematics. The concept of metric or M-s is a very important bridge in the transition from classical analysis to modern analysis. Because it allows us to carry operations in real or complex spaces to any space. Now, let's start by giving the concept of M-s [2] Jungck studied FXP theorems for commutative transformations in full M-ss and Jungck's work was developed by Fisher and other mathematicians [3]. FXP theory studies are not only limited to the full metric and normed spaces, but also ordered Banach spaces, regular, etc. It has also been; completed space. FXP theory is widely used in the theory of differential equations, integral equations, partial differential equations and related areas. FXP theory, like eigenvalue problems, boundary value problems, and approximation problems, has a lot of applications. In recent years, FXP theory studies have been carried over to the partial M-s, which is larger than the M-ss, and better results have been obtained than the results obtained in the M-s. Unlike the Caristi theorem, the completeness of the partial M-s is also characterized by Acar (2012) [4].

Purpose of the study

James Caristi [5] proved the following theorem in an article he published in 1976:

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"Let be (X,d) a full M-s and $\varphi: X \to \mathbb{R}$ an inferiorly bounded and inferiorly semi-continuous function defined by$T: X \to X$, for each $x \epsilon X$

$$d(x,Tx) \leq \varphi(Tx)$$

Let be a transformation that satisfies its inequality. In this case it has a FXP. Later, this theorem of Caristi was generalized by many authors, its applications were made and its proofs were made in different spaces [6]. This document is intended to demonstrate and generalise the Caristi fixed-point theorem in partial M-s.

Discussion

Matthews defines the partial metric concept on a non-empty set. One of the most important properties of partial M-s is that the distance between two points cannot be zero. Romaguera claims that Caristi's theorem in partial M-s is "necessary and sufficient for the partial M-s to be complete, if every defined -Caristi transformation has a FXP," starting with Kirk's Caristi type fixed-point theorem. I assumed it would take the form. However, he recognised that this was incorrect in the following example.

Example 1 [7]: A partial metric on the natural number set

Let $p(n,m) = max\{\frac{1}{n}, \frac{1}{m}\}$ on the set of $\mathbb{N}$ natural numbers be the partial metric p. The partial M-s $(\mathbb{N},p)$ is not complete. Because the metric p^s obtained with p forms the discrete topology on$\mathbb{N}$, and the $(n)_{n\epsilon\mathbb{N}}$ sequence $(\mathbb{N},p^s)$ is also a Cauchy sequence. But there is no defined p-Caristi transform onp^s. Indeed, $f: \mathbb{N} \to \mathbb{N}$ is defined and $\phi: (\mathbb{N},\tau_p) \to [0,\infty)$ is semi-continuous from below, and $p(n,fn) \leq \phi(n) - \phi(fn)$ for every$n \epsilon \mathbb{N}$. If $1 < f1$ then $p(1,f1) = 1 = p(1,1)$ which means $f1 \epsilon Bp(1,\varepsilon)\, for\, \epsilon > 0$. Like this ϕ, since it is semi-continuous from the bottom, $p(1) \leq p(f1)$, which contradicts $p(1,f1) \leq \phi(1) - \phi(f1)$. then $1 = f1$, which contradicts $p(1,f1) \leq \phi(1) - \phi(f1)$. That is, there is no p-Caristi transform defined on the set of $\mathbb{N}$ natural numbers.

Now let us give Romaguera in partial M-s to the FXP theorem of Caristi and its proof.

Theorem 1.1. (X,p) partial M-s 0- if and only if it is complete - p^s Caristi every transformation has a FXP.

Proof: Let us suppose that the partial M-s (X,p) is 0-complete and that T, X is the p^s-Caristi transform. Thus $p: X \to [0,\infty)$ has a bottom semi-continuous function, and for every $x \in X$

$$p(x,Tx) \leq p(x) - (Tx)$$

Condition is provided. Now for every $x \epsilon X$

$$A_x = \{y \in X: p(x,y) \leq \varphi(x) - \varphi(y)\}$$

Let's define the set. Since $T \in E$ is $Ax \neq \emptyset$ and Ax are closed. Let's take $x_0 \in X$. Let's choose $x^1 \in A_{xo}$ so that $\varphi(x_1) \leq inf_{y \in A_{xo}} \varphi(y) + 2^{-1}$.

Obviously $A_{x1} \subseteq A_x\, is$. Thus for every$x \epsilon X$.

$$p\,(x_1,x) \leq \varphi(x_1) - \varphi\,(x) \leq inf_{y \in A_{xo}} \varphi(y) + 2^{-1} - \varphi\,(x) \leq \varphi(x) + 2^{-1} - \varphi\,(x) \leq 2^{-1}$$

It is possible. Continuing this process, we create a $\{x_n\}$ array in X such that,

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i. For every $n \in N^+ \; and \; x_{n+1} \in A_{xn}$ for $A_{x_{n+1}} \subseteq A_{xn}$
ii. For every $n \in N$ and $x \;\; A_{xn}$ for $p\,(x_n, x) \leq 2^{-1}$

Conditions are provided. $p\,(x_n, x_n) \leq p\,(x_n, x_{n-1})$ and above i & ii. With the terms $\lim_{n,m\to\infty} p\,(x_n, x_n) = 0$ and $\{x_n\}$ is a 0-Cauchy sequence. As per our hypothesis,

$$\lim_{n,m\to\infty} p\,(z, x_n) = p\,(z, z) = 0$$

There is a $z \in X$ such that $\lim_{n\to\infty} p^s\,(z, x_n) 0$ in that case,

$$Z \in \bigcap_{n\in\mathbb{N}} A_{xn}$$

Is finally, let's show that $z = Tz$. For each $n \in \mathbb{N}$

$$p\,(x_n, Tz) \leq p\,(x_n, Z) + p\,(z, Tz) \leq \varphi(x_n) - \varphi(z) - \varphi(Tz)$$

it happens that

$$TZ \in \bigcap_{n\in\mathbb{N}} A_{xn}$$

Shows that. Thus(ii). It is obtained from $p\,(x_n, Tz) \leq 2^{-1}$.

$$p\,(z, Tz) \leq p\,(z, x_n) + p\,(x_n, Tz) \;\; and \;\; \lim_{n,m\to\infty} p\,(z, x_n) = 0$$

Given that $p\,(z, Tz) = 0$ is found. Thus $z = Tz$ is obtained.

Conversely, let's assume that $\{x_n\}$ 0-Cauchy sequence (X, p^s) is not convergent either. Subsequence $\{x_n\}$of $\{y_n\}$

$$p\,(y_n, y_{n+1}) < 2^{-(n+1)}$$

Let's create it. Let's take the set $A = \{y_n : n \in \mathbb{N}\}$ Let $T: X \to X$ be $Tx = y_o$ for transform $x \in X \setminus A$ and $n \in \mathbb{N}$ for every $T_{y_n} = T_{y_{n+1}}$. The closure of cluster A can be easily seen. Now let's define the transform $X \to [0, \infty)$ for $x \in X \setminus A$ as $\emptyset(x) = p(x, y_o) + 1$ and for each $n \in \mathbb{N}$ let it be $\emptyset(y_n) = 2^{-n}$

Like this,

$\emptyset(y_{n+1}) = 2^{-(n+1)} < 2^{-n} = \emptyset(y_n)$ and for every $x \in X \setminus A$

$$\emptyset(y_o) = 1 < p(x, y_o) + 1 = \emptyset(x)$$

It is possible. In this case $\emptyset$ is semi-continuous from the bottom. Also for each $x \in X \setminus A$

$$p(x, Tz) = p(x, y_o) = \emptyset(x) - \emptyset(y_o) = \emptyset(x) - \emptyset(Tx)$$

and for every $y_n \in A$

$$p\,(y_n, Ty_n) = p\,(y_n, y_{n+1}) < 2^{-(n+1)}$$

$$= \emptyset(y_n) - \emptyset(y_{n+1})$$

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$$= \emptyset(y_n) - \emptyset(Ty_n)$$

It is possible. So there is a Caristi transformation p^s that cannot have a FXP on (T, X) which is a inconsistency. Thus, the proof is completed.

2. Caristi Transformation in Partial M-s

As it is known, Romaguera Caristi gave the definition of Caristi transformation in two different ways in order to carry the FXP theorem to the partial M-s. But it should be noted that X is neither a p −Caristi transform nor a p^s-Caristi transform in partial M-s, although the unit transformation we get on it is a Caristi transform in M-s.

Let us now give the FXP theorem with the help of the Caristi transform in partial M-s.

Theorem 2.1: Every Caristi transformation defined on $'X'$ has a FXP, which is both a necessary and sufficient condition for the (X, p) partial M-s to be complete.

Proof: Let's assume that (X, p) is complete and T is a Caristi transform on X. In this case (X, p^s), there is a semi-continuous function from the bottom $\emptyset: X \to [0, \infty)$and for each $x \in X$

$p(x, Tx) \leq p(x, x) + \emptyset(x) - \emptyset(Tx)$

is provided. Like this

$2p(x, Tx) - p(Tx, Tx) \leq 2p(x, x) + 2\emptyset(x) - 2\emptyset(Tx) - p(Tx, Tx)$ (Eq.2.1)

inequality is achieved. Now let's define the $\beta: X \to [0, \infty)$, $\beta(x): X \to p(x, x)$function for each $x \in X$. Thus, $\beta, (X, p^s)$, is also semi-continuous from the bottom, and the function $\varphi = \beta + 2\emptyset$ is also semi-continuous from the bottom. Thus (2.1) inequality

$2p(x, Tx) - p(x, x) - p(Tx, Tx) \leq \varphi(x) - \varphi(Tx)$

If written in the form $2p(x, Tx) - p(x, x) - p(Tx, Tx) = p^s(x, Tx)$ is used

$$p^s(x, Tx) \leq \varphi(x) - \varphi(Tx)$$

obtained. Thus, from Lemma (X, p^s), is a full M-s and φ is semicontinuous from the bottom, T has a FXP from the Caristi FXP theorem.

Now let's show that X is complete if the T, Caristi transform has a FXP. Let's assume that X does not have a FXP in the full partial M-s T, and $\{x_n\}$ is a Cauchy sequence formed by the outliers of (X, p), and (X, p^s) does not converge $\{y_n\}$, the sequence of $\{x_n\}$.

$$p\,(y_n, y_{n+1}) - p\,(y_n, y_n) \leq 2^{-(n+1)}$$

Let be a subsequence that satisfies the condition. Consider the closed set $A = \{y_n : n \in \mathbb{N}\}$. Let's define the transformations of T and $\emptyset$ as $x \in X \setminus A$ and for each $n \in \mathbb{N}$ in the $T: X \to y_o$ form, and $Ty_n, y_{n+1}$and $(\emptyset: X \to [0, \infty), \emptyset(x) = p(y_o, x) + 1$, and $\emptyset(y_n) = 2^{-n}$. So for every $n \in \mathbb{N}, \emptyset\, y_{n+1} < \emptyset\, y_n$ and for every $x \in X \setminus A$ it becomes $\emptyset\,(y_o) \leq \emptyset(x)$. Considering Lemma it is seen that $\emptyset$ is also semi-continuous from the bottom (X, p^s). Also, for each $x \in X \setminus A$.

$$p(x, Tx) = p(x, y_o)$$

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$$= \emptyset(x) - \emptyset(y_o)$$
$$= \emptyset(x) + \emptyset(x) - \emptyset(Tx)$$

and for every $y_n \in A$

$$p\,(y_n, Ty_n) = p(y_n, y_{n+1})$$
$$\leq p(y_n, y_n) + 2^{-(n+1)}$$
$$= p\,(y_n, y_n) + +\emptyset(y_n) - \emptyset(Ty_n)$$

It is possible. This is in contradiction with the fact that T has no FXP. So our assumption is wrong that X has a FXP in the full partial M-s T.

Example 1.1: Let $X = [0,1]$ and for every $x, y \in X, p(x,y) = \max\{x,y\}$. Thus (X,p) is a full partial M-s and hence 0-complete. If $T: X \to X, Tx = \sqrt{x},\ and\ \emptyset: X \to [0,\infty), \emptyset(x) = 1 - x$ are defined as $\emptyset$, (X, p^s) they are also continuous and semi-continuous from the bottom. Also for each $x \in X$

$p(x,Tx) = \sqrt{x} \leq p\,(x,x) + \emptyset(x) - \ \emptyset(Tx)$

and T is a Caristi transform at X. Thus from Theorem 2.1, T has a FXP. However, it should be noted that Theorem 1.1 is not provided for this example. Because;

$$p(1,T1) = 1 \nleq 0 = \emptyset(1) - \ \emptyset(T1)$$

Such that, there cannot be a function $\emptyset: X \to \ [0,\infty)$.

3. Some generalisations in partial M-s of the Caristi type fixed-point theorem

In this section, the theorem of the FXP type Caristi will be given on partial M-s.

Theorem 3.1. Let (X,p) be the full partial M-s $p: X \to \ [0,\infty)$

$p(x,x) = (p(x,y) while\ \emptyset\ (y) \leq \emptyset(x)$ (1)

A semi-continuous function from below that satisfies the condition and $\varphi: X \to \ [0,\infty), \mu > 0$, including

$\sup\{\varphi(x): x \in X, \emptyset(x) \leq inf_{w \in X}\emptyset(x) + \mu\}$ (2)

Let be a function. If $T: X \to X$ transform for every $x \in X$

$p(x,Tx) \leq p(x,x) + \varphi(x)\{\varphi(x) - \varphi(Tx)\}$ (3)

If it satisfies the requirement, T has a FXP at X.

Proof. In case of $\varphi(x) > 0$, it is $\varphi(Tx) \leq \varphi(x)$ from (Eq.3) inequality. $\varphi(x) = 0$ is $p(x,Tx) = p(x,x)$, which is obtained from (1) inequality $\emptyset\big(Tx \leq \emptyset(x)\big)$. So for every $x \in X$, it is $\emptyset(Tx) \leq \emptyset(x)$. Now

$Y = \{x \in X: \emptyset(x) \leq inf_{w \in X}\emptyset(x) + \mu\}$

and as $\gamma = sup_{w \in X}\varphi(x) < \infty$. Since (X,p) is complete, $(X,p^s)$is also complete, and ϕ, $(X,p^s)$is semi-continuous from the bottom, set Y is closed. Thus (Y,p^s) is complete and

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(Y,p) is complete. Y is different from null and it is $\emptyset(Tx) \leq \emptyset(x)$ since it is $x \in X$ for every $TY \subseteq Y$. Also for each $x \in Y$

$p(x,Tx) \leq p(x,x) + \gamma\{\emptyset(x) - \emptyset(Tx)$ (4)

If $\varphi: Y \to [0,\infty)$is defined as $\varphi(x) = \gamma\varphi(x)$, then φ function (Y,p^s) is also semi-continuous from the bottom. Thus, according to **Theorem 2.1**, T has a FXP.

Theorem 3.2. Let (X,p) be a full partial M-s, $\varphi: X \to [0,\infty)$ $(Eq\ 1)$ a semi-continuous function from below and $c: [0,\infty) \to [0,\infty)$ a semi-continuous function from above, satisfying the condition Eq 1. If the transformation $T: X \to X$ for every$x \in X$.

$p(x,Tx) \leq p(x,x) + max\{c(\emptyset(x)), c(\emptyset(Tx))\}\{\emptyset(x) - \emptyset(Tx)\}$ (5)

Inequality, T , X also has a FXP.

Proof. Let $\gamma > c\ inf_{w\in X}\emptyset(w)$ be. c being semi-continuous from above,

$$t \in [inf_{w\in X}\emptyset(w), inf_{w\in X}\emptyset(w) + \mu\ while\ c(t) \leq inf_{w\in X}\emptyset(w) + \mu]$$

It has $\mu > 0$ to be. $\varphi: X \to [0,\infty)$ for each $x \in X$.

$$\varphi(x) = \max\{c(\emptyset(x)), c(\emptyset(Tx))\}$$

Let's define the format. As in *Theorem 3.1*, we can show that for every $x \in X$ there is $\emptyset(Tx) \leq \emptyset(x)$. So for $x \in X$

$\emptyset(x) \leq inf_{w\in X}\emptyset(w) + \mu$

Using this,

$\emptyset(Tx) \leq inf_{w\in X}\emptyset(w) + \mu$

is found. Thus $\emptyset(x) \leq \gamma$ is obtained. From here

$\sup\{\emptyset(x): x \in X, \{\emptyset(x) \leq inf_{w\in X}\emptyset(w) + \mu \leq \gamma < \infty$

and the desired result is obtained from *Theorem 3.1*.

Theorem 3.3. Let (X,p) be a full partial M-s, $\emptyset: X \to [0,\infty)$ a semi-continuous and $c: [0,\infty) \to [0,\infty)$ non-decreasing function satisfying the condition (Eq 1). If the transformation for each $x \in X$ is $T: X \to X$

$p(x,Tx) \leq p(x,x) + c(\emptyset(x))\{\emptyset(x) - \emptyset(Tx)\}$ (6)

Or

$p(x,Tx) \leq p(x,x) + c(\emptyset(Tx))\{\emptyset(x) - \emptyset(Tx)\}$ (7)

If it satisfies its conditions, T, X also has a FXP

Proof. As in **Theorem 3.1,** we can show that for every $x \in X$ there is $\emptyset(Tx) \leq \emptyset(x)$. Thus, since c is non-decreasing, $c(\emptyset(Tx) \leq c(\emptyset(x))$ Obtained.

So, it is sufficient to require the condition (Eq.7) and do the proof only for (Eq.6). If we define function, $\varphi: X \to [0,\infty)$ for $x \in X$, as $\varphi(x) \leq c(\emptyset(x))$

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$\sup\{\varphi(x): x \in X, \{\emptyset(x) \leq inf_{w\in X}\emptyset(w) + 1\} \leq c\left(inf_{w\in X}\emptyset(w) + 1\right) < \infty$

and the desired result is obtained from **Theorem 3.1.**

Theorem 4. Let (X, p) be a full partial M-s, $\emptyset: X \to [0, \infty)$, a semi-continuous function from the bottom and a semi-continuous function from the top $c: [0, \infty) \to [0, \infty)$, satisfying the condition (Eq1). If the $x \in X$ transformation for every $T: X \to X$

$p(x, Tx) \leq \emptyset(x)$ (8)

Or

$p(x, Tx) \leq p(x, x) + c\big(p(x, Tx)\big)\{\emptyset(x) - \emptyset(Tx)\}$ (9)

If it satisfies its conditions, T, X also has a FXP.

Proof. Let's define it as $\varphi: X \to [0, \infty), \varphi(x) = c(p(x, Tx))$ for $x \in X$

$$\emptyset(x) \leq inf_{w\in X}\emptyset(w) + 1$$

using that

$$\varphi\ (x) \leq \sup\{c(t): 0 \leq t \leq p(x, Tx)\}$$
$$\leq \sup\{c(t): 0 \leq t \leq \emptyset(x)\}$$
$$\leq \sup\{c(t): 0 \leq t \leq inf_{w\in X}\emptyset(w) + 1\}$$

and so since is semi-continuous from above,

$$\sup\{\varphi(x): x \in X, \emptyset \leq inf_{w\in X}\emptyset(w) + 1\}$$
$$\leq Max\{c(t): 0 \leq t \leq inf_{w\in X}\emptyset(w) + 1 < \infty$$

obtained. The desired outcome is thus derived from *Theorem 3.1*.

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