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STUDY THE FLEXIBILITY OFFERED IN THE CHOICE OF FEEDBACK GAIN MATRIX IN MULTIVARIABLE SYSTEMS

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Abstract

One of the most popular techniques for altering the response characteristics of a control system is the application of linear state variable feedback .The fact of using state feedback to assign the closed loop system self-conjugate set of eigenvalues, provided that the open loop system is controllable, is a well known and commonly used technique. For single-input systems, the state feedback gain is uniquely determined by the desired pole pattern, but for multiinput systems, there is a lot of freedom in choosing the parameters of the feedback gain matrix, which can assign a specified eigenvalue spectrum. There are few systematic ways of finding a unique gain feedback matrix, such as minimizing a quadratic performance index in which case the gain will be obtained by solving a Riccatti equation, or assigning both the eigenvalues and their eigenvectors .The main contribution of this paper is the elaboration of a new procedure for obtaining a unique state feedback gain matrix, which places the closed loop system poles to the desired locations and meets the following criteria: 1_Achieving the best possible time response characteristics, 2_Yielding a system with small feedback gains, 3_ Yielding a system with a good robustness.

Keywords: MIMO Systems, SISO Systems State Feedback, Gain Matrix, System Eigenvalues, Closed Loop Poles.

I. INTRODUCTION

One of the first applications of state-space methods to linear systems was that of using feedback of the state variables to relocate the eigenvalues of a given system [2, 6, 7]. J.Berhamin1959 was perhaps the first to realize [7] that if a given system realization was state controllable, then any desired characteristic polynomial could be obtained by state-variable feedback [2, 21].

In 1962, Rosenbrock discussed, among other things, the use of state feedback to relocate the eigenvalues of a plant so as to achieve better response characteristics, but a complete analysis was not pursued [15]. The same result was independently deduced in almost the same way by Popov in 1964 [18], who in fact treated the multi-input problem. In classical control system designs such as the root locus method, the objective is to adjust the feedback gains such that the closed-loop system has a desired pole pattern [19]. This is because the response is characterized to a large extent by the poles: In particular, stability of a linear system is determined by the locations of the poles, and the damping ratios of the fundamental modes determine the overshoot and the settling time for a step input. The same design considerations remain valid for multivariable problems [6, 7, 15]. Eigenvalues remain important in characterizing stability, and to a lesser extent, performance, and robustness [1, 5]. For a multivariable system, the response is

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characterized by both the eigenvalues and eigenvectors. This has motivated eigenstructure assignment design [1, 7, 11, 24]

The problem to be considered is as follows: Given a system described in state space formula

$$\dot{x}(t)= Ax(t)+ Bu(t) \qquad (1.1)$$

$$Y(t) = Cx(t) \qquad (1.2)$$

Where A, B, C are constant matrices of dimensions nxn, nxm, qxn respectively: $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control input vector. When and how can we apply feedback to this system, such that the resulting closed-loop system has a desired set of arbitrary poles or eigenvalues?

It is known that the closed-loop eigenvalues can be assigned to any self-conjugate set provided that the pair (A, B) is completely controllable. For single-input systems, the state feedback gain K is uniquely determined, by the desired pole pattern, and can be conveniently calculated, for example by Ackermann's formula [20]. For multinput systems there is a lot of freedom in choosing the parameters of K. Many researchers have utilized this design freedom to satisfy additional performance criteria.

The problem of pole assignment has been extensively studied by many researchers over the last decade [30]. It is simply concerned with moving the poles (or eigenvalues) of a given time invariant linear system to a specified set of locations in the s-plane (subjected to complex pairing), by means of state feedback or output feedback. The fact of using state feedback to assign the closed-loop system self-conjugate set of eigenvalues, provided that the open loop system is controllable, is a well-known and commonly used technique [2].

A large number of algorithms exist for the solution of the pole assignment problem [44]. Both the state-feedback methods and output-feedback methods results in compensator matrices which are, broadly speaking either dyadic (i.e. have rank equal to one) or have full rank. Although the dyadic algorithms have considerable elegance and simplicity, the resulting closed-loop systems have poor disturbance rejection properties compared with full-rank counterparts [26].

The rest of this paper is organized as follows: Section 2 deals with the needed notations, and section 3 presents the problem formulation. Next, Section 4 discusses the Controllability Matrix. Then, section 5 explains how to find the Controllability indices. Section 6 explains the proposed approach. To demonstrate the effectiveness of the proposed algorithm, practical example included in Section 6 in order to demonstrate the proposed approach. A brief discussion about the obtained results is given in section 7. Finally, Section 8 draws conclusions and future work.

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II. LIST OF SYMBOLS AND ACRONYMS

x(t)	State vector.
y(t)	Output vector.
u(t)	Input vector.
n	Number of system states.
m	Number of inputs.
$\Re$	Set of real numbers
w_n,w_μ,$\wp$,$\emptyset_c$	Controllability matrix.
k_i	Controllability indices.
∆A	Perturbation.
$\hat{m}$	Rank.
T_c, Q	Transformation matrix.
T_c^{-1}	Inverse of transformation matrix.
A_c,B_c	Multivariable controllable forms.
C_c	A general form.
0_m	Null matrices.
I_m	Identity matrices.
A_r,C_r	Block elements.
q	k_i×n matrix.
Ebc	Elementary matrix.
T_r	Rise time.
T_s	Settling time.
T_p	Peak time.
M_p	Maximum overshoot.
POS	Percent overshoot.
Ess	Steady-State Error.
K_c	Feedback gain matrix.
A_D	Closed-loop matrix.
α(s)	Characteristics polynomial.

III. PROBLEM STATEMENT

Given a multivariable system described by state space description.

$$\dot{x}(t)=Ax(t)+Bu(t) \qquad (3.1)$$

$$y(t)= cx(t) \qquad (3.2)$$

Where A, B, C are constant matrices of dimensions nxn, nxm, qxn respectively, and x(t) $\in R$, u(t)$\in R^m$, and a given set of desired eigenvalues, Z

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We want to find a nmxn gain matrix K such that under the state feedback operation,

$$u(t)=r(t)-Kx(t) \quad (3.3)$$

The new state equation

$$\dot{x}(t)=(A-BK)x(t)+Br(t) \quad (3.4)$$

Has the desired eigenvalues and meets the following criteria:

1- Achieving the best possible closed-loop system time response characteristics (small transient, small steady state error)

2- Yielding a system with small feedback gains to avoid saturation of the control elements.

3- Yielding a system with a good robustness, that is the close-loop system should be less sensitive to variations or perturbations of the closed loop system matrix parameters.

To solve the problem, we will assume that the given system should be completely state controllable.

IV. CONTROLLABILITY MATRIX

The system described by equations (1.1) is said to be controllable if one of the following conditions given by the following theorem is satisfied:

Theorem 4.1:

The necessary and sufficient condition for the system (1. 1) to be completely controllable is given by one of the following conditions:

(i) W(0,t_1)= $\int_0^t e^{-At}BB^Te^{-A^Tt}\,dt$ is non-singular,

The controllability matrix

$$\delta = [B, AB, A^2B \dots A^{n-1}B]_{nxnm} \text{ rank } \delta = n$$

Proof see [6].

The following corollary is very useful in computer implementation. The dimension of the controllability matrix is considerably reduced.

Corollary 4.1:

Any system identical to that described by the above equations is controllable if the (n x (n-k+l) m) matrix:

W_{n-k}=[B AB A ^{2}B … A $^{n\ k}$B] has a full rank, where k = rank (B) [6].

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V. CONTROLLABILITY INDICES

Considering the system as described by equations (2.1), its controllability matrix is therefore given by

$$W_n = [BABA^2B\ldots A^{n-1}B] \quad (5.1)$$

And

$$B = [b1\ b2\ b3\ \ldots\ b_m] \quad (5.2)$$

Where bi is a column vector of dimension n x 1, W_n can be rewritten as:

$$W_n = [b_1 b_2 \ldots b_m;\ Ab_1; Ab_2 \ldots Ab_m;\ \ldots;\ A^{n-1}b_1 \ldots A^{n-1}b_m] \quad (5.3)$$

Searching linearly independent columns of W_n from left to right in order, which means a column is linearly dependent if it can be rewritten as a linear combination of its left hand-side columns, otherwise it is linearly independent.

At the end of this search process, we will get a set of linearly independent columns after leaving out the linearly dependent ones. Taking this set of vectors and rearranging them, we get the following ordered set:

$$\{b_1 Ab_1 \ldots A^{K_1-1}b_1; b_2 AB_2 \ldots A^{K_2-2}b_2; \ldots; b_m AB_m \ldots A^{k_m-1}b_m\} \quad (5.3)$$

Where the integer ki represents the number of linearly independent columns associated with bi in the above vector set.

The controllability index is defined as the maximum integer ki in the following set:

$$\{k_1 k_2 k_3 \ldots k_m\} \quad (5.4)$$

The set $\{k_1 k_2 k_3 \ldots k_m\}$ is called the set of controllability indices of the system. If the system is controllable then $k_1 + k_2 + k_3 + \ldots + k_m = n$.

The controllability indices will play an important role in the design of state feedback as it will be seen later in the coming chapter.

V.1.Canonical forms

Consider the MIMO system given by

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (5.5a)$$

$$Y(t) = Cx \quad (5.5b)$$

It is required to determine a matrix T such that transformation.

$$X = Tz, \quad z = T^1 X$$

Produces a canonical form of the system (2.7)

$$\dot{z}(t) = A_0 z(t) + B_0 u(t) \quad (5.6a)$$

$$Y(t) = C_0 z(t) \quad (5.6b)$$

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Formal definition of the term 'canonical' can be found in the papers of Popov (1972) and Wang and Davison (1976). A survey of Luenberg forms is due to Sinha and Rosa (1976). In most cases it is assumed that the system (5.5) is completely controllable so that the stander condition (Baenett 1975) holds, namely that the controllability condition matrix

$$\delta(\mathrm{A, B}) = [\mathrm{B, AB}, A^2B, . . . A^{n-1}B] \qquad (5.7)$$

Has full rank.

VI. PROPOSED APPROACH

The contribution of thesis is concerned with the use of the freedom offered in the choice in the choice of the gain feedback matrix in state feedback design of multivariable systems beyond specification of the closed loop eigenvalues. The given system described in state space is first transformed into general controller companion form, and then a study is conducted to determine a unique feedback gain matrix, which assigns the closed loop eigenvalues to desired locations and meets the following criteria:

1) Achieving the best possible closed loop system time response characteristics.
2) Yielding a system with small feedback gains to avoid saturation of the control elements.
3) Yielding a system with a good robustness.

The step response is arguably the most relevant test of general system performance and as such it is worth discussing some of the related criteria which are used to specify system performance.

Accepted dynamic performance measure relate to the degree of oscillation, the time for the transient to die away and the response time or time for the response to reach some fraction of the final values.

Transient Response Specification

In many practical cases, the desired performance characteristics of control systems are specified in terms of time domain quantities.

Frequently, the performance characteristics of a control system are specified in terms of the transient response to a unit step input.

The performance evaluation is done in terms of the following quantities:

1) Maximum overshoot M_p
2) Rise time T_r
3) Peak time T_p
4) settling time T_s
5) steady state error

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The aforementioned criteria are defined as follows:

1. Maximum overshoot M_p is the magnitude of the first overshoot. It may also be expressed in percent of the final value, that is

$$\text{Percent maximum overshoot} = \frac{\textbf{maximum overshoot}}{\textbf{final value}} \times 100\%$$

2. Time to maximum overshoot T_p is the time required to reach the maximum overshoot
3. Settling time T_s is the time required for the output response first to reach and thereafter remain within 5% or 2% of the final value, or it is the smallest value T_s such that

$$|y(t) - y_s| \leq 0.02\, y_s \text{ Or } \quad 0.05 y_s \quad \text{for all } t \geq T_s$$

4. Rise time T_r is defined as the time for the response on its initial rise, to go from 0.1 to 0.9 times the steady-sate value. Approximately T_r is such that y (T_r)$\cong 0.9y_s$.

Hence, the response characteristics, such as maximum overshoot and settling time can be compared for different values of the gain matrix K.

Proposed Procedure

As a first step, the given n-dimensional matrices A and B are converted into multivariable controllable companion forms A_c and B_c , if the system is completely controllable[2], Where A_c and B_c have the forms

$$A_c = \begin{pmatrix} A_{11} & A_{12} & . & . & . & A_{1m} \\ A_{21} & A_{22} & . & . & . & A_{2m} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ A_{m1} & A_{m2} & . & . & . & A_{mm} \end{pmatrix}, \; B_c = \begin{pmatrix} B_1 \\ B_2 \\ . \\ . \\ . \\ B_m \end{pmatrix} \qquad (6.1)$$

And C_c is in general form. The block matrices, A_{ii}, A_{ij} and B_i are respectively of dimensions $k_i x k_i$, $k_i x k_j$ and k_ix m where k_i is a controllability index and $\sum_{i=1}^{m} k_i$=n. The block matrices are of the form given below:

$$A_{ii} = \begin{pmatrix} 0 & 1 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 1 \\ x & x & . & . & . & x \end{pmatrix}, A_{ij} = \begin{pmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \\ x & x & . & . & . & x \end{pmatrix},$$

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$$B_i = \begin{pmatrix} 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & x & . & . & . & x \end{pmatrix} \qquad (6.2)$$

Where x is a non-trivial element and the first i-1 columns of the matrix B_i are zero.

It is possible to write the controllable matrix A_c-B_cK_c in many different ways depending on the choice of the feedback gain matrixK_c; we may have one or more back diagonal elements, where each block is in companion form. To illustrate the procedure, we give the following example. In order not to be overwhelmed by notation, we assume n=6 and m=3. It is also assumed that the controllability indices are k_1= 2, k_2= 2 and k_3= 2. Then we have

$$A_c = T_c A T^{-1}{}_c = \begin{bmatrix} 0 & 1 & | & 0 & 0 & | & 0 & 0 \\ x & x & | & x & x & | & x & x \\ - & - & | & - & - & | & - & - \\ 0 & 0 & | & 0 & 0 & | & 0 & 0 \\ x & x & | & x & x & | & x & x \\ - & - & | & - & - & | & - & - \\ 0 & 0 & | & 0 & 0 & | & 0 & 1 \\ x & x & | & x & x & | & x & x \end{bmatrix}$$

$$\text{and} \begin{bmatrix} 0 & 0 & 0 \\ 1 & x & x \\ - & - & - \\ 0 & 0 & 0 \\ 0 & 1 & x \\ - & - & - \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (6.3)$$

The pair $\{A_c, B_c\}$ is said to be in a multivariable controller form.

The introduction of u = r-$K_c x_c$ a 3x6 real constant matrix yields

$$x_c = (A_c - B_cK_c)x_c + B_c r \qquad (6.4)$$

Because of the form of B_c, all rows of A_c-B_cK_c expect the three rows containing strings of x, are not affected by state feedback. Because the three nonzero rows of B_c are linearly independent, the three rows of A_c- B_cK_c containing strings of x can be affected by the state feedback gain matrixK_c.

Different choices of the number of blocks and their sizes will lead to a number of possible cases.

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Case 1: we choose K_c so that (A_c-B_cK_c) is a single block in companion form, of dimension (k_1+k_2+k_3) x(k_1+k_2+k_3)

$$\begin{bmatrix} 0 & 1 & | & 0 & 0 & | & 0 & 0 \\ 0 & 0 & | & 1 & 0 & | & 0 & 0 \\ - & - & | & - & - & | & - & - \\ 0 & 0 & | & 0 & 1 & | & 0 & 0 \\ 0 & 0 & | & 0 & 0 & | & 1 & 0 \\ - & - & | & - & - & | & - & - \\ 0 & 0 & | & 0 & 0 & | & 0 & 1 \\ -d_1 & -d_2 & | & -d_3 & -d_4 & | & -d_5 & -d_6 \end{bmatrix} \quad (6.5)$$

The characteristics polynomial in this case is:

$$A_d(s) = s^6 + d_6s^5 + d_5s^4 + d_4s^3 + d_3s^2 + d_2s + d_1 \quad (6.6)$$

Case 2: we choose K_c so that (A_c-B_cK_c) has 2 block of companion forms on the diagonal, with dimensions (k_1+k_2) x ($k_1 + k_2$) and (k_1xk_3) respectively.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & | & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 0 \\ 0 & 0 & 0 & 1 & | & 0 & 0 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & | & 0 & 0 \\ - & - & - & - & | & - & - \\ 0 & 0 & 0 & 0 & | & 0 & 1 \\ 0 & 0 & 0 & 0 & | & -c_1 & -c_2 \end{bmatrix} \quad (6.7)$$

The characteristics polynomial is:

$A_d(s) = (s^4 + \alpha_4 s^3 + \alpha_3 s^2 + \alpha_2 S + \alpha_1)\ (s^2 + c_2 s + c_1)$ (6.8)

and $C = \begin{bmatrix} 0 & 1 & -12 & 0.80 \\ 0 & 1 & -2.5 & 0 \end{bmatrix}$

We want to set the closed loop poles -8; -9; - and -6.50. Since rank $\Phi_c = [BABA^2BA^3B] = 4$, the pair $(A,\ B)$ is completely state controllable. Thereforethe pair $(A,\ B)$ can be converted to multivariable controllable companion form $(A_B,\ B_c)$. In this example controllability indices are $k_1 = 2$ and $k_2 = 2$. The pair $(A_c,\ B_c)$ and C_c are as follows:

Case 3: we choose K_c so that (A_c−B_cK_c) has 2 block of companion forms on the diagonal, with dimension (k_1xk_1) and (k_2+k) (k_2+k_3) respectively.

$$\begin{bmatrix} 0 & 1 & | & 0 & 0 & 0 & 0 \\ -\alpha_1 & -\alpha_2 & | & 0 & 0 & 0 & 0 \\ - & - & | & - & - & - & - \\ 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & | & 0 & 0 & 0 & 1 \\ 0 & 0 & | & -\beta_1 & -\beta_2 & -\beta_3 & -\beta_4 \end{bmatrix} \quad (6.9)$$

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The characteristic polynomial is:

$$A_d(s)= (s^2+ a_2 s + a_1)(s^4 + \beta_4 s^3 + \beta_3 s^2 + \beta_2 s + \beta_1) \qquad (6.10)$$

Case 4: we choose K_c so that (A_c-$B_c K_c$) has 3 blocks of companion forms on the diagonal, with dimension (k_1xk_1), (k_2xk_2) and (k_3xk_3) respectively.

$$\begin{bmatrix} 0 & 1 & | & 0 & 0 & | & 0 & 0 \\ -a_1 & -a_2 & | & 0 & 0 & | & 0 & 0 \\ - & - & | & - & - & | & - & - \\ 0 & 0 & | & 0 & 1 & | & 0 & 0 \\ 0 & 0 & | & -b_1 & -b_2 & | & 0 & 0 \\ - & - & | & - & - & | & - & - \\ 0 & 0 & | & 0 & 0 & | & 0 & 1 \\ 0 & 0 & | & 0 & 0 & | & -c_1 & -c_2 \end{bmatrix} \qquad (6.11)$$

The characteristic polynomial is:

$$A_d(s)= (s^2 a_2 s + a_1)\,(s^2 + b_2 s + b_1)\,(s^2 + c_2 s + c_1) \qquad (6.12)$$

There is great deal of freedom in the choice ofK_c, because we can define serval different matrices (A_c−$B_c K_c$) with the same characteristic polynomial α(s). However this freedom of choice in the selection of the gain matrix should ideally be used to achieve some other desirable properties of the closed-loop system besides selecting the closed loop eigenvalues

For every chosen structure of (A_c−$B_c K_c$) (one, two, three, or more companion – from blocks on the diagonal), the step response of the closed loop system

$$\dot{x_c}(t)= (A_c - B_c K_c)x_c(t) + B_c r(t) \qquad (6.13\ )$$

y= $C_c x_c$(t)

is plotted and the time response characteristics

(Maximum overshoot, settling time, Rise time, Peak time, and steady -state error) computed. The above results are then compared to select a feedback gain matrix K_c meeting the required criteria (small transient response, small feedback gain, good

Illustrative Example

Consider the system below given its matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 4 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 13.00 & -3.00 & -5.00 & 3.00 \\ 0 & 0 & 0 & 1 \\ -9.00 & -7.00 & 3.00 & 2.00 \end{bmatrix}$$

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$$B_c = T_c B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } C_c = \begin{bmatrix} -0.60 & 0.80 & 7.60 & 1.80 \\ 2.50 & 0 & 0.50 & 1.00 \end{bmatrix}$$

It then follows that:

$$A_c - B_c K_c = A_c - Ebc\overline{K} = A_D \begin{Bmatrix} Adesiredclosed - loopmatrixwith\ the \\ givensetofdesiredeigvalues \end{Bmatrix}.$$

Where $K_c = D^{-1}\overline{K}$ and the original feedback gain matrix given by

$K = K_c T_c$.

$$Ebc = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$ And the matrix D is chosen such that

$$B_c = EbcD.$$

The matrix D is found to be:

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Ebc\overline{K} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \overline{K}_{13} & \overline{K}_{12} & \overline{K}_{13} & \overline{K}_{14} \\ 0 & 0 & 0 & 0 \\ \overline{K}_{21} & \overline{K}_{22} & \overline{K}_{23} & \overline{K}_{24} \end{bmatrix}$$

$$A_c - Ebc\overline{K} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 13.00 - \overline{K}_{11} & -3.00 - \overline{K}_{12} & -5.00 - \overline{K}_{13} & 3.00 - \overline{K}_{14} \\ 0 & 0 & 0 & 1 \\ -9.000\overline{K}_{21} & -7.000 - \overline{K}_{22} & 3.00 - \overline{K}_{23} & 2.00 - \overline{K}_{24} \end{bmatrix}$$

$$= A_D \begin{Bmatrix} A\ desired\ closed - loop\ matrix\ with\ the \\ given\ set\ of\ desired\ eignvalues \end{Bmatrix}$$

With different combinations of the controllability indices k_i for$(i = 1,\ 2)$, two cases can arise.

<u>Case1:</u> when r=1, that is we have just one block of companion form with order $k_1 = 4$, the desired characteristic polynomial and the desired closed-loop matrix A_D are as follows:

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$$\alpha(s) = s^4 + 31.50s^3 + 370.50s^2 + 1928s + 3744A_D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3744 & -1928 & -370.50 & -31.50 \end{bmatrix}$$

The required feedback gain matrix in the original coordinate system for this case is

$$k_1 = \begin{bmatrix} 4.00 & 10.00 & -1.00 & -7.00 \\ 166.00 & -1721.50 & -2527.50 & 1755.00 \end{bmatrix}$$

The following simulation results are obtained for this case:

Table 1: Results of the Simulation Obtained for r=1

Transient & Steady- State Specifications	M_P	E_{ss}	POS	t_r	t_s	t_p
Output y_1	0.81	0.00	80.87	1	7	3
Output y_2	0.33	0.00	29.59	3	10	5

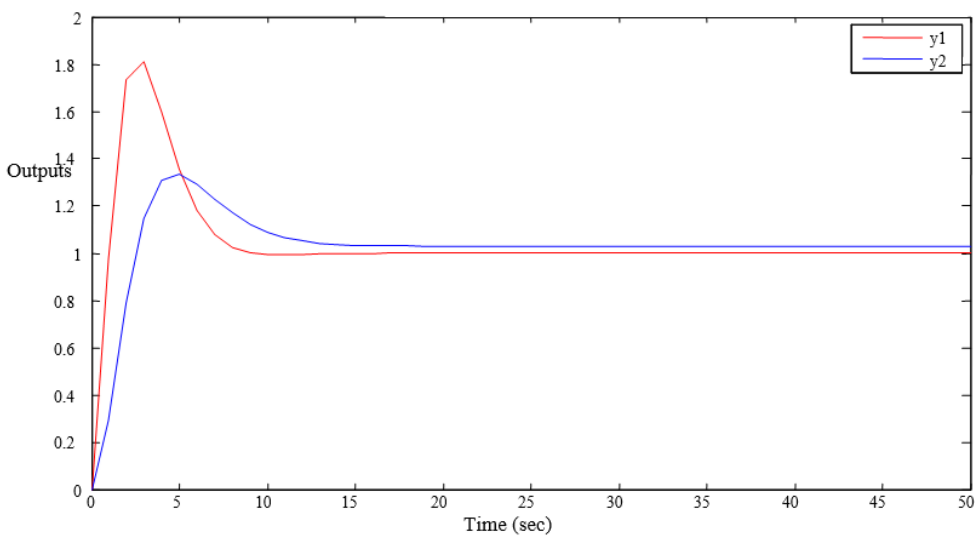

Figure 1: Step Response of the Outputs y_1(t) and y_2(t)

<u>case 2</u>: when r=2, that is we have 2 blocks of companion forms with orders $k_1 = 2$ and $k_2 = 2$ respectively, the desired characteristic polynomial and the desired closed-loop system matrix A_D are as follows:

$$\alpha(s) = (s^2 + 17s + 72)(s^2 +)14.50s + 52$$

$$A_D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -72 & -17 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -52 & -14.50 \end{bmatrix}$$

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The required feedback gain matrix in the original coordinate system for this case

$$k_{22} = \begin{bmatrix} 43.00 & 32.00 & -58.00 & -29.00 \\ 10.50 & 34.00 & -91.50 & -17.50 \end{bmatrix}$$

The following simulation results are obtained for this case:

Table 2: Results Of the Simulation Obtained for r=2

Transient & Steady- State Specification	M_P	E_{ss}	POS	t_r	t_s	t_p
Output y_1	0.24	0.00	24.48	1	6	2
Output y_2	1.81	0.00	0.00	5	6	NO

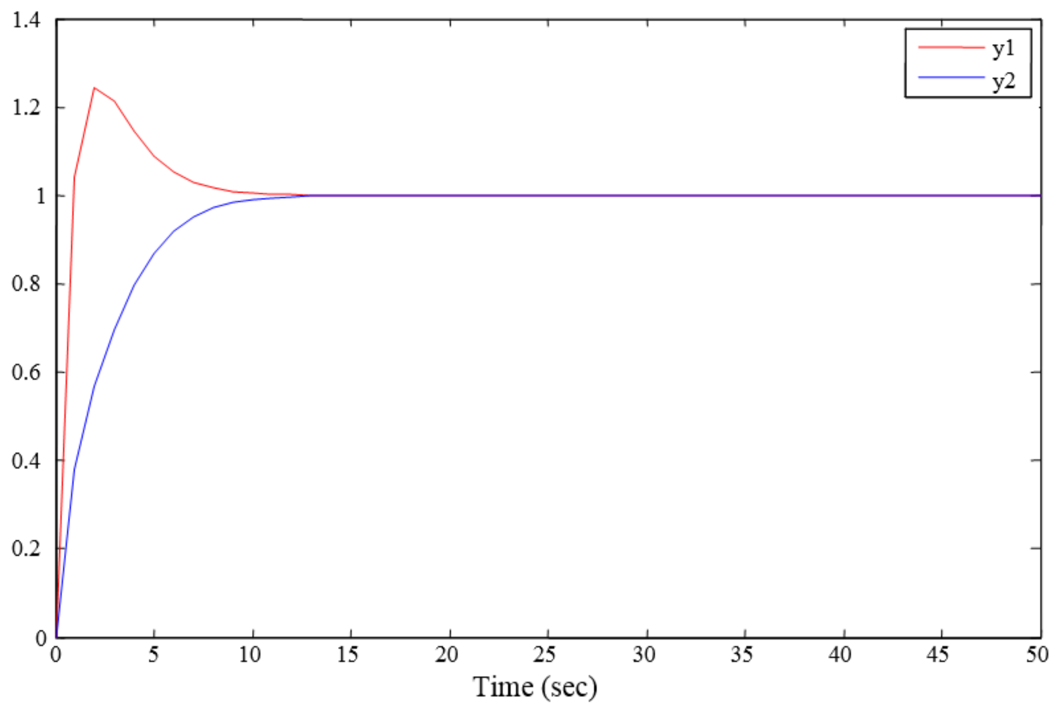

Figure 2: Step Response of the Outputs $y_1(t)$ and $y_2(t)$

For the sensitivity of closed-loop system subjected to the following random perturbation

$$\Delta A = \begin{bmatrix} 0.05 & 0.01 & 0 & 0.03 \\ 0.04 & -0.06 & 0.01 & 0 \\ 0 & 0.03 & 0.02 & 0.01 \\ -0.04 & -0.04 & 0.05 & 0.02 \end{bmatrix}$$

The following simulation results are obtained for each case.

<u>For case 1</u>: when r=1, we have obtained the following results.

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Table 3 : Results of the Simulation Obtained for r=1

Transient & Steady- State Specifications	E_{ss}		POS	t_r	t_s	t_p
Output y_1	0.66	0.33	146.80	1	8	2
Output y_2	0.07	0.67	49.05	2	6	4

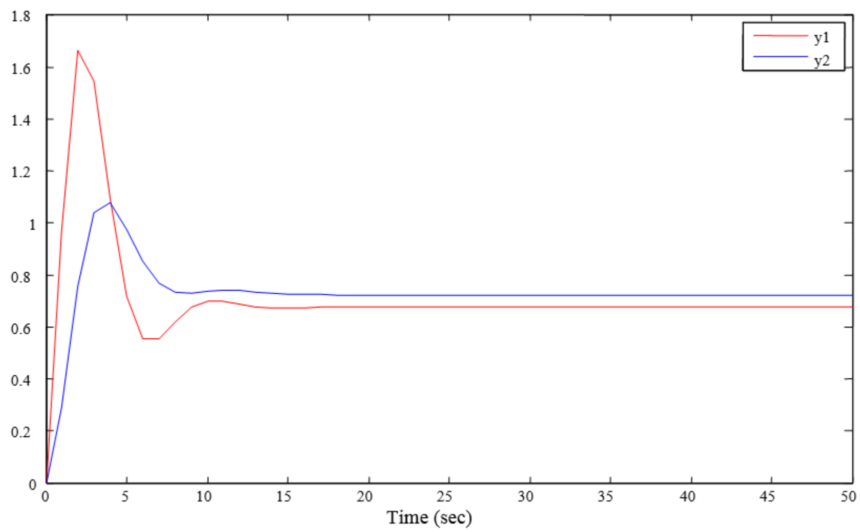

Figure 3: Step Response of the Outputs$y_1(t)$ and $y_2(t)$

For case2: when r=2, that is we have 2 blocks of companion form on the diagonal with orders $k_1 = 2$ and $k_2 = 2$ respectively.

We have obtained the following simulation results.

Table III. 4 : Results of the Simulation for r=2

Transient & Steady- State Specifications	M_P	E_{ss}	POS	t_r	t_s	t_p
Output y_1	0.21	0.02	24.98	1	5	2
Output y_2	0.00	0.00	0.00	6	7	NO

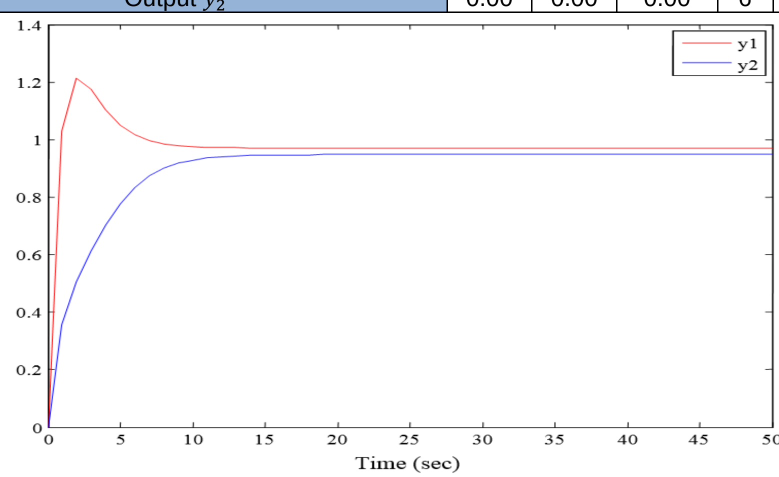

Figure 4: Step Response of the output$y_1(t)$ and$y_2(t)$

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Summary of Simulation

The following tableau is a summary of the simulation results obtained for this example:

Table 5 : Summary of Simulation

		One block withorder4	Two blocks with orders $k_2 = 1$ and $k_2 = 2$
y_1	M_P	0.81	0.24
	E_{ss}	0.00	0.00
	Pos	80.87	24.48
	t_r	1	1
	t_s	7	6
	t_p	3	2
y_2	M_P	0.33	0.00
	E_{ss}	0.00	0.00
	Pos	29.59	0.00
	t_r	3	5
	t_s	10	6
	t_p	5	No

VII. DISCUSSIONS AND INTERPRETATION OF THE RESULTS

The feedback gain matrix that results in the greatest number of blocks of companion forms on the closed-loop system matrix's diagonal ($A_c - B_c K_c$) represents the best approach under consideration, the feedback gain matrix that best places the closed-loop system's eigenvalues in the desired positions and best satisfies the stated desired criteria:

- ✓ Best possible time response: The gain matrix is optimized to achieve the best possible time response for the control system. This involves minimizing settling time, rise time, overshoot, and other performance metrics.
- ✓ Small feedback gains: The gain matrix is designed to have small individual gain values. This is desirable because it reduces the sensitivity of the closed-loop system to measurement noise and disturbances. Smaller gains also help in mitigating control effort and actuator saturation issues.
- ✓ Insensitivity to small random perturbations: The gain matrix is chosen to make the closed-loop system less sensitive to small random perturbations or modeling uncertainties. This improves the robustness and stability margins of the control system.

VIII. CONCLUSION

Multivariable system means a system that involves multiple variables that are interdependent and affect the behaviour of the others. These systems are widespread in various fields, including engineering. The analysis and understanding of multivariate systems is crucial to making informed decisions and accurately predicting their behaviour. Interactions between variables can be non-linear, and small changes in one variable can have significant effects on others. This complexity often requires sophisticated mathematical tools and computational techniques to effectively study and analyze these systems. The behaviour of these systems often depends on time, which means that variables change over time due to internal and external influences. The study of the flexibility of selecting the feedback gain matrix in multivariate systems is an important aspect in control theory and system design. Also, it is observed from the simulation results that the magnitude of the dynamical mode is decreased as the number of blocks of companion forms on the diagonal of is increased. However, we get less overshoot, hence less settling time which gives rise to better time response. In addition, it is noticed that the norm of the left eigenvector (for i=l, . . ., n) decreased as the number of companion blocks on the diagonal of the matrix () is increased. Thus, the eigenvalues of the closed loop system are less sensitive to perturbations in the case we have a maximum number of blocks of companion form on the diagonal of the closed loop system matrix.

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