Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
Journal of Tianjin University Science and Technology
ISSN (Online): 0493-2137
E-Publication: Online Open Access
Vol:55 Issue:01:2022
DOI 10.17605/OSF.IO/5NX69

ON SIX PLATONIC GRAPHS

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Abstract

We have computed here the γ_t refereed as total domination number of the six platonic graphs. Incidentally we found that the octahedron graph is a counter example to the following result: If G is a connected graph of order at least two, then $\gamma_t(G) \geq ecc(C(G)) + 1$ which appeared in [1].

Keywords: Domination set (DS), total domination set (TDS), Domination number (DN), total domination number (TDN), Platonic Graphs.

1 Introduction

Let G = (V, E) be a (p, q) graph with p = |V(G)| and q = |E(G)|. We adopt the standard notations for graph theoretic terms as per Bondy and Murty [2]. The distance d(u,v) between u and v in G is the length of a least u-v path in G. The eccentricity ecc(v) of v in G is the distance between v and a vertex at farthest from v in G. The minimum (maximum) eccentricity among the elements of V(G) is called the radius(diameter) of G and is denoted by rad(G)(diam(G)). The center C(G) of G is the set of all vertices of least eccentricity and the periphery B(G) is the set of all vertices of greatest eccentricity. A set P ⊆ V(G) is called a DS of G if every vertex in V-P is adjacent to some vertex in D. The least number of elements in a DS of G is called the DN $\gamma(G)$ of G. A DS P is called a TDS if the subgraph induced by P has no vertex of degree 0. The TDN of G is the least number of elements in a TDS of G and is denoted by $\gamma_t(G)$ [2].we define P ⊆ V(G) and v in P, then pn(v, P) = { w ∈ V| N(w)∩P = {v}}, ipn(v, P) = pn(v, P) ∩ P and epn(v, P) = pn(v, P) \ P.

2 Six Platonic Graphs

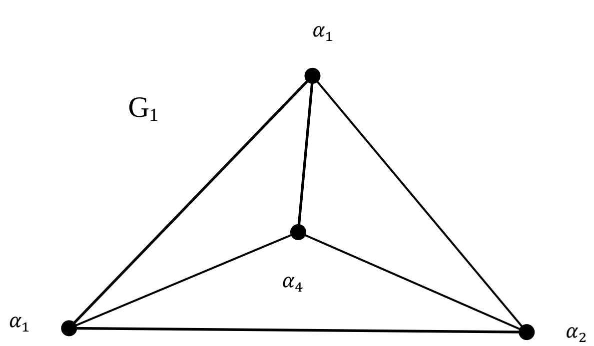

Fig 1(a) Tetrahedron Graph

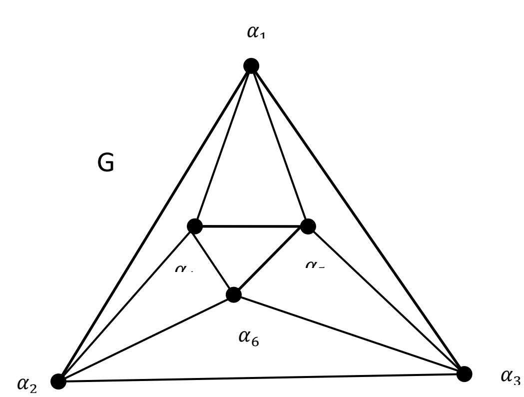

Fig 1(b) Octahedron graph

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G_3

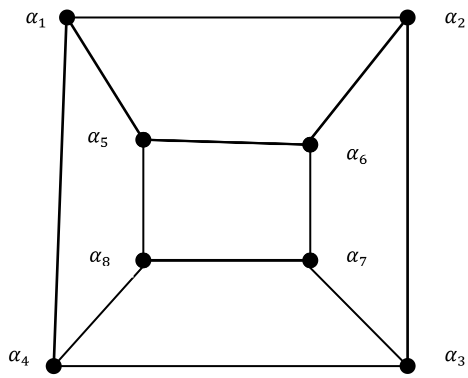

Fig 1(c) Hexahedron graph

G_4

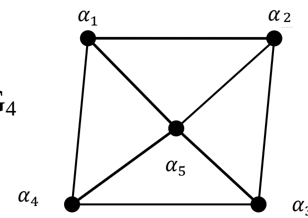

Fig 1(d) The Square Pyramid graph

G

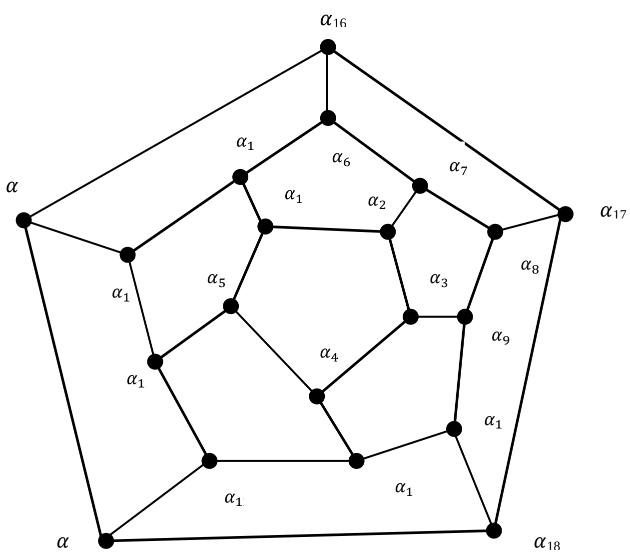

Fig 1(e) The Dodecahedron graph

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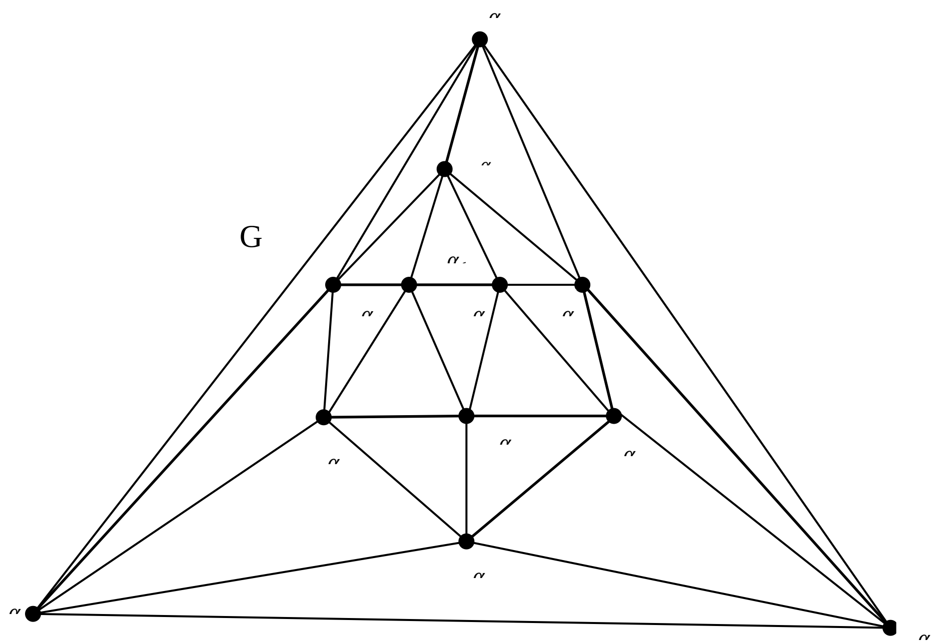
Fig 1(f) The Icosahedron graph



3. Total domination number of six Platonic Graphs

Theorem 3.1 $\gamma_t(G_1) = 2$

Proof: Consider the graph G_1. Let $V(G_1) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $E(G_1) = \{(\alpha_1, \alpha_2), (\alpha_1, \alpha_3), (\alpha_1, \alpha_4), (\alpha_2, \alpha_3), (\alpha_2, \alpha_4) (\alpha_3, \alpha_4)\}$. We see from Theorem 2 and Theorem 8 of Table 1 that $2 \leq \gamma_t (G_1) \leq 2$. Let P = $\{\alpha_1, \alpha_2\}$. Then P is a DS as $(\alpha_1, \alpha_3), (\alpha_1, \alpha_4)$ are edges of G_1. Also $(\alpha_1, \alpha_2) \in E(G_1)$ implies P is a TDS of G_1. As P – $\{\alpha_1\}$ and P – $\{\alpha_2\}$ are not TDS, we conclude that P is a minimal TDS.

Note 3.1.1 As one can find more minimal TDS of G_1 like $\{\alpha_1, \alpha_3\}$, $\{\alpha_1, \alpha_4\}$, $\{\alpha_2, \alpha_3\}$ we can say that the minimum TDS for G_1 does not exist.

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Statements of Well Known Results	G_1	G_2	G_3
1.If G is connected with p ≥3, then $\gamma_t(G) \leq {}^{2p}/_{3}$ [3].	$\gamma_t(G_1) \leq (2 \times 4)/3$ = 2.67	$\gamma_t(G_2) \leq (2 \times 6)/3$ = 4	$\gamma_t(G_3) \leq \frac{2 \times 8}{3}$ = 5.33
2.If G(p, q) has maximum degree atmost 3 and of order p and size q, then $\gamma_t(G) \leq p - {}^{q}/_{3}$ [4].	$\gamma_t(G_1) \leq 4 - \frac{6}{3} = 2$	—	$\gamma_t(G_3) \leq 8 - \frac{12}{3} = 4$
3. If G has no vertex of degree 0, then $\gamma_t(G) \geq {}^{p}/_{\Delta(G)}$ [3].	$\gamma_t(G_1) \geq {}^{4}/_{3}$ = 1.33	$\gamma_t(G_2) \geq {}^{6}/_{4}$ = 1.5	$\gamma_t(G_3) \geq {}^{8}/_{3}$ = 2.67
4. If G(p,q) with maximum degree at most n-2,then $\gamma_t(G) \leq p - \Delta(G)$.	–	$\gamma_t(G_2) \leq 6 - 4$ = 2.	$\gamma_t(G_3) \leq 8 - 3$ = 5
5. If G(p, q) graph with p ≥2, then $\gamma_t(G) \geq rad(G)$[1].	$\gamma_t(G_1) \geq 1$	$\gamma_t(G_2) \geq 2$	$\gamma_t(G_3) \geq 3$
6. Let G(p,q) with p ≥ 2 and let P be a $\gamma_t$set. Then necessary and Sufficient for $\gamma_t(G) = rad(G)$ is G[P] has $rad(G)/2$ edges [1] .	–	$\gamma_t(G_2) = 2.$	–
7. If G(p, q), with p ≥ 2 is connected then $(diam(G) + 1)/2 \leq \gamma_t(G)$ [1].	$(1 + 1)/2 \leq \gamma_t(G_1)$ = 1	$(2 + 1)/2 \leq \gamma_t(G_2)$ = 1.5	$(3 + 1)/2 \leq \gamma_t(G_3)$ = 2
8.If G(p, q), with p ≥ 2 is connected, then $\gamma_t(G) \geq ecc(C(G) + 1$ [1].	$\gamma_t\ (G_1) \geq 2$	–	$\gamma_t\ (G_3) \geq 4$
9.If G(p, q) with p ≥ 2 is connected, then $(3\ ecc(B) + 2)/4 \leq \gamma_t\ (G)$ [5]	$\frac{(3 \times 1) + 2}{4} \leq \gamma_t\ (G_1)$ = 1.25	$\frac{(3 \times 2) + 2}{4} \leq \gamma_t\ (G_2)$ = 2	$\frac{(3 \times 3) + 2}{4} \leq \gamma_t\ (G_3)$ = 2.75
10.If G(p, q) is graph of girth g, then $g/2 \leq \gamma_t(G)$ [1].	${}^{3}/_{2} \leq \gamma_t\ (G_1)$ = 1.5	${}^{3}/_{2} \leq \gamma_t\ (G_2)$ = 1.5	${}^{4}/_{2} \leq \gamma_t\ (G_3)$ = 2
11.If G(p, q) is connected with girth g ≥ 3 and with $\delta(G) \geq 2$, then $\frac{p}{2} + \max\left(1, \frac{p}{2(g+1)}\right) \geq \gamma_t$ [6].	$\frac{4}{2} + max\left(1, \frac{4}{2(3+1)}\right) \geq \gamma_t\ (G_1)$ = 3	$\frac{6}{2} + max\left(1, \frac{6}{2(3+1)}\right) \geq \gamma_t\ (G_2)$ = 4	$\frac{8}{2} + max\left(1, \frac{8}{2(4+1)}\right) \geq \gamma_t\ (G_3)$ =5
12.If G(p, q) is connected with minimum at least two and girth g ≥ 3, then $\left(\frac{1}{2} + \frac{1}{g}\right)p \geq \gamma_t$ [7].	$\left(\frac{1}{2} + \frac{1}{3}\right)4 \geq \gamma_t\ (G_1)$ = 3.35	$\left(\frac{1}{2} + \frac{1}{3}\right)6 \geq \gamma_t\ (G_2)$ = 5	$\left(\frac{1}{2} + \frac{1}{4}\right)8 \geq \gamma_t\ (G_3)$ = 6

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13.For every G(p, q) graph with no vertex of degree 0, $\gamma(G) \leq \gamma_t(G) \leq 2\,\gamma(G)$ [8].	$\gamma(G_1) = 1 \leq \gamma_t\,(G_1) \leq 2\gamma(G_1)$ = 2	$\gamma(G_2) = 2 \leq \gamma_t\,(G_2) \leq 2\gamma(G_1)$ =4	$\gamma(G_3) = 3 \leq \gamma_t\,(G_3) \leq 2\gamma(G_3)$ = 6
14.If G(p, q) is connected with $\delta(G) \geq 2$,then $\left\lfloor \frac{4}{7}\,(n+1) \right\rfloor \geq \gamma_t$ [9].	$\left\lfloor \frac{4}{7}\,(4+1) \right\rfloor \geq \gamma_t\,(G_1)$ = 3	$\left\lfloor \frac{4}{7}\,(6+1) \right\rfloor \geq \gamma_t\,(G_2)$ = 4	$\left\lfloor \frac{4}{7}\,(8+1) \right\rfloor \geq \gamma_t\,(G_3)$ = 5
15.If G(p, q) is connected with $\delta(G) \geq 3$, then $n/2 \geq \gamma_t$ [9].	$4/2 \geq \gamma_t\,(G_1) = 2$	$6/2 \geq \gamma_t\,(G_2) = 3.$	$8/2 \geq \gamma_t\,(G_3) = 4$
16.If a planar graph G with diam(G) = 2, then the domination number $\gamma(G)$ is at most 3. [10].	–	$\gamma(G_2) = 2 \leq 3$	-
17.If a planar graph G with diam(G) =2, then the total domination number $\gamma_t(G)$ is at most 3.[10].	–	$\gamma_t(G_2) = 2 \leq 3$	–

Table 1 Upper and lower bounds of $\gamma_t(G_i)$ for i = 1, 2, 3 with reference to various structural parameters.

Statements of Well Known Results	$\mathbf{G_4}$	$\mathbf{G_5}$	$\mathbf{G_6}$
1.If G is connected with p ≥3, then $\gamma_t(G) \leq {}^{2p}/_{3}$ [3].	$\gamma_t(G_4) \leq (2 \times 5)/3$ = 3.33	$\gamma_t(G_5) \leq (2 \times 20)/3$ = 13.33	$\gamma_t(G_6) \leq \frac{2 \times 12}{3}$ = 8
2. If G(p, q) has maximum degree atmost 3 and of order p and size q, then $\gamma_t(G) \leq p - {}^{q}/_{3}$ [4].	—	—	–
3. If G has no vertex of degree 0, then $\gamma_t(G) \geq {}^{p}/_{\Delta(G)}$ [3].	$\gamma_t(G_4) \geq {}^{5}/_{4}$ = 1.25	$\gamma_t(G_5) \geq {}^{20}/_{3}$ = 6.67	$\gamma_t(G_6) \geq {}^{12}/_{5}$ = 2.4
4. If G(p, q) with maximum degree at most n-2,then $\gamma_t(G) \leq p - \Delta(G)$.	–	$\gamma_t(G_5) \leq 20 - 3$ = 17	$\gamma_t(G_6) \leq 12 - 5$= 7
5. If G(p, q) graph with p ≥2, then $\gamma_t(G) \geq rad(G)$[1].	$\gamma_t(G_4) \geq 1$	$\gamma_t(G_5) \geq 5$	$\gamma_t(G_6) \geq 3$
6. . Let G(p, q) with p ≥ 2 and let P be a γ_t set. Then necessary and Sufficient for $\gamma_t(G) = rad(G)$ is G[P] has $rad(G)/2$ edges [1] .	–	–	–

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7.If G(p, q),with p ≥ 2 is connected then $(diam(G)+1)/2 \le \gamma_t(G)$ [1].	$(2+1)/2 \le \gamma_t(G_4)$ = 1.5	$(5+1)/2 \le \gamma_t(G_5)$ = 3	$(3+1)/2 \le \gamma_t(G_6)$ = 2
8. .If G(p, q), with p ≥ 2 is connected,then $\gamma_t(G) \ge ecc(C(G)+1$ [1].	$\gamma_t(G_4) \ge 2$	$\gamma_t(G_5) \ge 6$	$\gamma_t(G_6) \ge 4$
9.If G(p, q) with p ≥ 2 is connected, then $(3\, ecc(B)+2)/4 \le \gamma_t(G)$ [5]	$\frac{(3\times 2)+2}{4} \le \gamma_t(G_4)$ = 2	$\frac{(3\times 5)+2}{4} \le \gamma_t(G_5)$ = 4.25	$\frac{(3\times 3)+2}{4} \le \gamma_t(G_6)$ = 2.75
10.If G(p, q) is graph of girth g, then $g/2 \le \gamma_t(G)$ [1]	$^3/_2 \le \gamma_t(G_4)$ = 1.5	$^4/_2 \le \gamma_t(G_5)$ = 2	$^3/_2 \le \gamma_t(G_6)$ = 1.5
11. If G(p, q) is connected with girth g ≥3 and with $\delta(G) \ge 2$, then $\frac{p}{2}+\max\left(1,\frac{p}{2(g+1)}\right) \ge \gamma_t$ [6].	$\frac{5}{2}+max\left(1,\frac{5}{2(3+1)}\right) \ge \gamma_t(G_4)$ = 3.5	$\frac{20}{2}+ max\left(1,\frac{20}{2(4+1)}\right) \ge \gamma_t(G_5)$ = 12	$\frac{12}{2}+ max\left(1,\frac{12}{2(3+1)}\right) \ge \gamma_t(G_6)$ =7.5
12.If G(p, q) is connected with minimum at least two and girth g ≥ 3, then $\left(\frac{1}{2}+\frac{1}{g}\right)p \ge \gamma_t$ [7].	$\left(\frac{1}{2}+\frac{1}{3}\right)5 \ge \gamma_t(G_4)$ = 4.16	$\left(\frac{1}{2}+\frac{1}{4}\right)20 \ge \gamma_t(G_5)$ = 15	$\left(\frac{1}{2}+\frac{1}{3}\right)12 \ge \gamma_t(G_6)$ = 10
13. For every G(p, q) graph with no vertex of degree 0, $\gamma(G) \le \gamma_t(G) \le 2\,\gamma(G)$ [8].	$\gamma(G_4)=1 \le \gamma_t(G_4) \le 2\gamma(G_4)$ = 2	$\gamma(G_6)=6 \le \gamma_t(G_5) \le 2\gamma(G_5)$ =12	$\gamma(G_6)=3 \le \gamma_t(G_6) \le 2\gamma(G_6)$ = 6
14. If G(p, q) is connected with $\delta(G) \ge$ 2,then $\left\lfloor\frac{4}{7}(n+1)\right\rfloor \ge \gamma_t$ [9].	$\left\lfloor\frac{4}{7}(5+1)\right\rfloor \ge \gamma_t(G_4)$ = 3	$\left\lfloor\frac{4}{7}(20+1)\right\rfloor \ge \gamma_t(G_5)$ = 12	$\left\lfloor\frac{4}{7}(12+1)\right\rfloor \ge \gamma_t(G_6)$ = 7.42
15.If G(p, q) is connected with $\delta(G) \ge 3$, then $n/2 \ge \gamma_t$ [9].	$5/2 \ge \gamma_t(G_4) = 2.5$	$20/2 \ge \gamma_t(G_5) = 10$	$12/2 \ge \gamma_t(G_6) = 6$
16.If a planar graph G with diam(G) = 2, then the domination number $\gamma(G)$ is at most 3. [10].	$\gamma(G_4)=1 \le 3$	—	-
17. If a planar graph G with diam(G) =2, then the total domination number $\gamma_t(G)$ is at most 3.[10].	$\gamma_t(G_4)=2 \le 3$	—	–

Table 2 Upper and lower bounds of $\gamma_t(G_i)$ for i = 4, 5, 6 with reference to various structural parameters.

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Theorem 3.2 $\gamma_t(G_2) = 2$.

Proof Consider the graph G_2. Let V(G_2) = $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ and E(G_2) = { (α_1, α_2), (α_1, α_3), (α_1, α_4), (α_1, α_5), (α_2, α_3), (α_2, α_4), (α_2, α_6), (α_3, α_5), (α_3, α_6), (α_4, α_5), (α_4, α_6), (α_5, α_6)}. We see from Theorem 4 and Theorem 5 of Table 1 that 2 ≤ $\gamma_t\,(G_2)$ ≤ 2. Hence γ_t(G_2) = 2. Let P = { α_1, α_2}.Then P is a DS as (α_1, α_3) , (α_1, α_4), (α_1, α_5), (α_2, α_6) are edges of G_2. Also (α_1, α_2) ∈ $E(G_2)$ implies P is a TDS. As P – { α_1} and P- { α_2} are not TDS, we conclude that P is a minimal TDS.

Note 3.2.1 As one can find more minimal TDS of G_2 like $\{\alpha_1, \alpha_3\}$$\{\alpha_1, \alpha_4\}$,$\{\alpha_2, \alpha_3\}$ we can say that the minimum TDS for G_2 does not exist.

Theorem 3.3 $\gamma_t(G_3) = 4$.

Proof Consider the graph G_3. Let V(G_3) = $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$ and E(G_3)= { (α_1, α_2), (α_1, α_4), (α_1, α_5), (α_2, α_3), (α_2, α_6), (α_3, α_4), (α_3, α_7), (α_4, α_8) (α_5, α_6), (α_5, α_8), (α_6, α_7), (α_7, α_8)}.We see from Theorem 2 and Theorem 8 of Table 1 that 4 ≤ $\gamma_t\,(G_3)$ ≤ 4. Hence γ_t(G_2) = 4. Let P = { α_1, α_2, α_7, α_8}. Then P is a DS as (α_1, α_4) , (α_1, α_5), (α_2, α_3), (α_2, α_6) are edges of G_3. Also (α_1, α_2), (α_7, α_8) ∈ $E(G_3)$. Therefore P is a TDS. Now
G[P-{α_1}] contains isolated vertex u_2; G[P-{α_2}] contains isolated vertex u_1, G[P-{α_7}] contains isolated vertex u_8; G[P-{α_8}] contains isolated vertex u_7. So we conclude that P is a minimal TDS.

Note 3.3.1 As one can find more minimal TDS of G_3 like $\{\alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ we can say that the minimum TDS for G_3 does not exist.

Theorem 3.4 $\gamma_t(G_4) = 2$.

Proof Consider the graph G_4. Let V(G_4) = $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \}$ and E(G_4) = { (α_1, α_2), (α_1, α_4), (α_1, α_5), (α_2, α_3), (α_2, α_5), (α_3, α_4), (α_3, α_5), (α_4, α_5)}.We see from Theorem 8 and Theorem 13 of Table 2 that 2 ≤ $\gamma_t\,(G_1)$ ≤ 2. Hence γ_t(G_4) = 2. Let P = { α_1, α_2}. Then P is a DS as (α_1, α_4), (α_1, α_5), (α_2, α_3) are edges of G_4. Also (α_1, α_2) ∈ $E(G_4)$. Therefore P is a TDS. Now P- { α_2} and P – { α_1} are not TDS. So we conclude that P is a minimal TDS.

Note 3.4.1 As one can find more minimal TDS of G_4 like { α_1, α_4}, { α_1, α_5}we can say that the minimum TDS for G_4 does not exist.

Theorem 3.5 Let P be a TDS in a graph G. Then P is a minimal TDS if and only if |epn(v,P)| ≥1 or | ipn(v,P)| ≥1for every v in P [4].

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Theorem 3.6 $\gamma_t(G_5) = 8$.

Proof

Let $V(G_5) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{19}, \alpha_{20}\}$ and $E(G_5) = \{(\alpha_1,\alpha_2), (\alpha_1,\alpha_5), (\alpha_1,\alpha_{15}), (\alpha_2, \alpha_3), (\alpha_2, \alpha_7), (\alpha_3,\alpha_4), (\alpha_3,\alpha_9), (\alpha_4,\alpha_5),(\alpha_4, \alpha_{11}) (\alpha_5, \alpha_{13}), (\alpha_6,\alpha_7), (\alpha_6,\alpha_{15}), (\alpha_6, \alpha_{16}), (\alpha_7, \alpha_8), (\alpha_8, \alpha_9), (\alpha_8, \alpha_{17}), (\alpha_9, \alpha_{10}),(\alpha_{10}, \alpha_{11}), (\alpha_{10}, \alpha_{18}), (\alpha_{11},\alpha_{12}),(\alpha_{12}, \alpha_{13}), (\alpha_{12}, \alpha_{19}), (\alpha_{13}, \alpha_{14}), (\alpha_{14}, \alpha_{15}), (\alpha_{14}, \alpha_{20}), (\alpha_{16}, \alpha_{17}), (\alpha_{16}, \alpha_{20}),(\alpha_{17}, \alpha_{18}), (\alpha_{18}, \alpha_{19}), (\alpha_{19}, \alpha_{20})$. Consider the set $P = \{\alpha_2, \alpha_4, \alpha_7, \alpha_8, \alpha_{10}, \alpha_{11}, \alpha_{14}, \alpha_{20}\}$. As $(\alpha_1, \alpha_2),(\alpha_3, \alpha_4), (\alpha_4, \alpha_5), (\alpha_6, \alpha_7), (\alpha_8, \alpha_9), (\alpha_{11}, \alpha_{12}), (\alpha_{13}, \alpha_{14}), (\alpha_{14}, \alpha_{15}), (\alpha_{16}, \alpha_{20}),(\alpha_8, \alpha_{17}),(\alpha_{10}, \alpha_{18}), (\alpha_{19}, \alpha_{20})$ in $E(G_5)$ we TDS see that P is a DS of G_5. Moreover $(\alpha_2, \alpha_7), (\alpha_4, \alpha_{11}), (\alpha_7, \alpha_8), (\alpha_{10}, \alpha_{11}), (\alpha_{14}, \alpha_{20})$ in $E(G_5)$. So P is a TDS. Now we claim that P is a minimal TDS of G_5. This can be easily seen from the fact that P-{w}eitherviolates dominance property or violates the total dominance property for all w in P. Hence we conclude that P is a minimal TDS. We can also double check this assertion by verifying the following fact. Fact : If P is a minimal TDS of a graph G with n $\geq$ 3 vertices then every element α of P satisfies one of the two criteria given below: 1) there exist a w in V-P such that N(w) –P = $\{\alpha\}$ 2) The subgraph induced by P- $\{\alpha\}$ includes in it an isolated vertex. It is easy to verify that $\alpha_2, \alpha_4, \alpha_8, \alpha_{10}, \alpha_{14}, \alpha_{20}$ satisfies the criteria 1 and α_7, α_{11} satisfies the criteria 2. Hence $\gamma_t(G_5) = 8$.

Note 3.6.1 As $\{\alpha_1, \alpha_{2,}\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{16}, \alpha_{17}, \alpha_{20}\}$ is an alternative minimal TDS of G_5, we conclude that the minimum TDS does not exist for G_5.

Theorem 3.7 $\gamma_t(G_6) = 4$.

Proof Consider the graph G_6. Let $V(G_6) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}\}$ and $E(G_6) = \{(\alpha_1, \alpha_2), (\alpha_1, \alpha_3), (\alpha_1, \alpha_4), (\alpha_1, \alpha_5), (\alpha_1, \alpha_8), (\alpha_2, \alpha_3), (\alpha_2, \alpha_5), (\alpha_2, \alpha_9), (\alpha_2, \alpha_{12}), (\alpha_3, \alpha_8), (\alpha_3, \alpha_{11}), (\alpha_3, \alpha_{12}), (\alpha_4, \alpha_5), (\alpha_4, \alpha_6), (\alpha_4, \alpha_7), (\alpha_4, \alpha_8), (\alpha_5, \alpha_9), (\alpha_5, \alpha_6), (\alpha_6, \alpha_7), (\alpha_6, \alpha_9), (\alpha_6, \alpha_{10}), (\alpha_7, \alpha_8), (\alpha_7, \alpha_{10}), (\alpha_7, \alpha_{11}), (\alpha_8, \alpha_{11}), (\alpha_9, \alpha_{10}), (\alpha_9, \alpha_{12}), (\alpha_{10}, \alpha_{11}), (\alpha_{10}, \alpha_{12}), (\alpha_{11}, \alpha_{12})\}$. We see from Theorem 8 and Theorem 13 of Table 2 that $4 \leq \gamma_t(G_6) \leq 6$. Let $P = \{\alpha_2, \alpha_3, \alpha_6, \alpha_7\}$.Then P is a DS as (α_1, α_2), (α_4, α_6), (α_2, α_5), (α_7, α_8), (α_6, α_9), (α_6, α_{10}), (α_7, α_{11}), (α_2, α_{12}) are edges of G_6. Also $(\alpha_2, \alpha_3), (\alpha_6, \alpha_7) \in E(G_6)$. Therefore P is a TDS. We know from Theorem 3.5 [4] that P is a minimal TDS. This is because, $pn(\alpha_2,P) = \{\alpha_3\}$ as $N(\alpha_3) \cap P = \{\alpha_2\}$implies ipn (α_2, P) = pn $(\alpha_2, P) \cap P = \{\alpha_3\}$. Similarly ipn$(\alpha_3,P) = \{\alpha_2\}$; ipn$(\alpha_6, P) = \{\alpha_7\}$; ipn$(\alpha_7, P) = \{\alpha_6\}$. Hence | ipn (α, P) | $\geq$ 1 for every α in P of G_6. Hence P is a minimal TDS of G_6.

Note 3.7.1 As $\{\alpha_2, \alpha_3, \alpha_5, \alpha_6\}$ is an alternative minimal TDS of G_6 the minimum TDS does not exist for G_6.

4. Counter Example

Note that the Octahedron G_2 is a counter example for Theorem 8 [1] which says : " If G is a connected graph of order at least 2, then $\gamma_t(G_2) \geq ecc(C(G) + 1$". In G_2, note that C(G_2) = $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ as all vertices of G_2 have equal eccentricity. So ecc(C(G_2)) = max $\{d_{G_2}(\alpha, C(G_2))$: $\forall\ \alpha \in V(G_2)\}$ = 2. This implies $\gamma_t(G_2) \geq 2 + 1 = 3$, a contradiction to the fact that $\gamma_t(G_2) = 2$ indicated by Theorem 4 and Theorem 5 stated in Table.1.

5. Conclusion

For all the six platonic graphs we have verified the various bounds (both lower and upper) for the TDN provided by 17 different results of various authors found in the literature. we have found the exact TDN for all the six platonic graphs. Also we have incidentally found a counterexample to one of the results obtained by DeLaVina et.al [1].

Acknowledgement

The second author (Yegnanarayanan Venkataraman) acknowledge the National Board of Higher Mathematics, Department of Atomic Energy, Government of India, Mumbai for financial support by their grant no. 02011/10/21NBHM- (R.P)/R&D-II/8007/Date:13-07-2021 The first author acknowledges Kalasalingam Academy of Research and Education for its financial support by means of a Research Fellowship.

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