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DEGREE FACTORIAL ENERGY OF SOME GRAPHS

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Abstract

In this paper, we introduce a new energy of a graph called degree factorial energy and obtained the degree factorial energy of path graph, cycle graph, grid graph, wheel graph, star graph and triangular snake graph.

Keywords: Degree Factorial Matrix, Degree Factorial Spectrum and Degree Factorial Energy.

1.INTRODUCTION

Let $G(V,E)$ be a simple graph with n vertices and m edges. The adjacency matrix $A(G)$ of a graph G is a square matrix of order n whose $(i,j)^{th}$ entry is equal to 1 if the vertex v_i is adjacent to the vertex v_j and is equal to 0 otherwise $[1]$. The spectrum of the graph is defined as the set of all eigen values of the adjacency matrix and the energy of the graph is defined as the sum of all absolute values of these eigen values $[3,4]$. There are many energies based on matrices like distance matrix, Laplacian Matrix $[2,4,5,7,8,9]$ etc., Erich Huckel initiated the application of graph theory in chemistry during the 1930s when he explored molecular orbital theory[6]. He devised a graphical framework for unsaturated hydrocarbons in quantum chemistry, transforming it into a graph theory problem. Remarkably, the energy levels of electrons in these unsaturated hydrocarbon molecules were found to correspond to the Eigen values of the associated graphs, which led to the naming of this concept. In our contemporary understanding, this approach remains a fundamental connection between graph theory and chemistry. In this paper the degree factorial energy of the graph is defined and the same is obtained for path graph, cycle graph, complete graph, ladder graph, wheel graph, star graph and triangular snake graph.

2.MAIN RESULTS

Definition 2.1 Let $G(V,E)$ be a simple graph. Let $V=\{v_1,v_2,\dots,v_n\}$. The degree factorial matrix of the graph G is denoted by $DFM(G)=[a_{ij}]$ and is defined as follows:

$$DFM(G)=\begin{cases} d(v_i)! & if\ i=j \\ 0! & otherwise \end{cases}.$$

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The characteristic polynomial of $DFM(G)$ is $P_{DFM(G)}(\lambda) = \det(DFM(G) - \lambda I)$ is known as degree factorial polynomial of G where I is the identity matrix. The roots of the equation $P_{DFM(G)}(\lambda) = 0$ are called as the degree factorial eigen values of G (or) degree factorial characteristic values of G. The set of all eigen values $\lambda_1, \lambda_2, \dots\dots, \lambda_n$ of this polynomial $P_{DFM(G)}(\lambda)$ is known as the degree factorial spectrum of G. The sum of the absolute values of the degree factorial eigen values of G is known as the degree factorial energy of G and is denoted by $DFM(G)$. $i.e., DFM(G) = \sum_{i=1}^{n} |\lambda_i|$.

Theorem 2.1. The degree factorial energy of the path graph P_n is $2(n-1)$ where $n \geq 2$.

Proof: Let P_n be a graph with n vertices and $n-1$ edges. The degree sequence of the path graph P_n is $(1, 2, 2, \dots, 1) = (1,1, 2, 2, \dots, 2)$. Let $v_1, v_2, \dots, v_n$ be the corresponding vertex labels of the above degree sequence of the path graph P_n.

Now consider the path graph P_2. The degree factorial matrix of P_2 is

$$DFM(P_2) = \begin{pmatrix} 1! & 1 \\ 1 & 1! \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The characteristic equation of $DFM(P_2)$ is

$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda = 0$. The degree factorial spectrum and the degree factorial energy of P_2 graph are $0, 2$ and 2 respectively.

For $n \geq 3$, the degree factorial matrix of P_n consists of 4 blocks where the blocks

$$B_1 = \begin{bmatrix} d(v_i)! & 1 \\ 1 & d(v_i)! \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \end{bmatrix}_{2\times(n-2)},$$

$$B_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix}_{(n-2)\times 2} \quad \text{and } B_4 = \begin{bmatrix} 2! & 1 & 1 & \cdots & 1 \\ 1 & 2! & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 2! \end{bmatrix}_{(n-2)\times(n-2)}$$

The above blocks can be written as $B_1 = J_{2\times 2}$, $B_2 = J_{2\times(n-2)}$, $B_3 = J_{(n-2)\times 2}$ and $B_4 = I_{(n-2)\times(n-2)} + J_{(n-2)\times(n-2)}$. (Here, $I_{n\times n}$ is the identity matrix and $J_{n\times n}$ is the matrix of ones)

$$i.e., DFM(P_n) = \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right].$$

Now the characteristic equation of $DFM(P_n)$ is

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$$\begin{vmatrix} 1!-\lambda & 1 & 1 & 1 & . & . & . & 1 \\ 1 & 1!-\lambda & 1 & 1 & . & . & . & 1 \\ 1 & 1 & 2!-\lambda & 1 & . & . & . & 1 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & . & . & . & 2!-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda-1)^{n-3}(\lambda^2-(n+1)\lambda+2)=0.$$

∴ The degree factorial spectrum of the path graph P_n is 0, $\frac{n+1\pm\sqrt{(n+1)^2-8}}{2}$, $(n-3)$ times 1.

Now the degree factorial energy of the path graph P_n is

$$= \underbrace{0+1+1+\cdots\ldots+1}_{n-3\ times} + \left|\frac{n+1\pm\sqrt{(n+1)^2-8}}{2}\right| + \left|\frac{n+1\pm\sqrt{(n+1)^2-8}}{2}\right|$$

$$= n-3+\left[\frac{n+1\pm\sqrt{(n+1)^2-8}}{2}\right] + \left[\frac{n+1\pm\sqrt{(n+1)^2-8}}{2}\right]$$

$$= n-3+\frac{n+1}{2}+\frac{n+1}{2}$$

$$= 2n-2 = 2(n-1).$$

Hence the degree factorial energy of the path graph P_n is $2(n-1)$ where $n \geq 2$.

Theorem 2.2. The degree factorial energy of the cycle graph C_n is $2n$ where $n \geq 3$.

Proof:

Let $C_n, n \geq 3$ be a cycle graph with n vertices and n edges. The degree sequence of the cycle graph C_n is $(2, 2, \ldots, 2)$. Let $v_1, v_2, \ldots, v_n$ be the corresponding vertex labels of the above sequence. The degree factorial matrix of C_n is

$$DFM(C_n) = [2!\, I_{n\times n} + J_{n\times n} - I_{n\times n}]$$

where $I_{n\times n}$ is the identity matrix and $J_{n\times n}$ is the matrix of ones.

The characteristic equation of $DFM(C_n)$ is

$$\begin{vmatrix} 2!-\lambda & 1 & 1 & 1 & . & . & . & 1 \\ 1 & 2!-\lambda & 1 & 1 & . & . & . & 1 \\ 1 & 1 & 2!-\lambda & 1 & . & . & . & 1 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & . & . & . & 2!-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)^{n-1}\big(\lambda-(n+1)\big)=0$$

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The degree factorial spectrum of the cycle graph C_n is $n+1$, $(n-1)$ times 1.

Now the degree factorial energy of the cycle graph is

$$DFM(C_n) = n + \underbrace{1 + 1 + 1 + \cdots . +1}_{(n-1)\ times} = n + 1 + n - 1 = 2n.$$

Theorem 2.3. The degree factorial energy of the complete graph $K_n, n \geq 2$ is $n!$.

Proof:

Let K_n be a complete graph with n vertices. The degree sequence of complete graph K_n is $(n-1, n-1, \dots, n-1)$. Let $v_1, v_2, \dots, v_n$ be the corresponding vertex labels of the above sequence. The degree factorial matrix of K_n is

$$DFM(K_n) = \left[\left((n-1!) - 1\right) I_{n\times n} + J_{n\times n}\right]$$

The characteristic equation of $DFM(K_n)$ is

$$\begin{vmatrix} (n-1)! - \lambda & 1 & \dots & 1 \\ 1 & (n-1)! - \lambda & \dots & 1 \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ 1 & 1 & \dots 1 & (n-1)! - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - [(n-1)! - 1])^{n-1}\left(\lambda - (n-1)! + (n-1)\right) = 0$$

The degree factorial spectrum of the complete graph K_n is $(n-1)$ times $(n-1)! - 1$, $(n-1)! + (n-1)$.

Now the degree factorial energy of the complete graph K_n is

$$\begin{aligned} DFE(K_n) &= (n-1)! + (n-1) + \left((n-1)! - 1\right)(n-1) \\ &= (n-1)[(n-2)! + 1 + (n-1)! - 1] \\ &= (n-1)!\,[(n-2)! + (n-1)!] \\ &= (n-1)! + (n-1)(n-1)! \\ &= n! \end{aligned}$$

Hence the degree factorial energy of the complete graph K_n is $n!$.

Theorem 2.4. The degree factorial energy of grid graph with $P_2 \times P_n$ is $4(3n-4)$.

Proof:

Let $P_2 \times P_n, n \geq 2$ be a grid graph with 2n vertices and $3n-2$ edges. The degree sequence of grid graph $P_2 \times P_n$ is $(2, 2, 2, 2, 3, 3, \dots, 3, 3)$. Let $v_1, v_2, \dots, v_n$ be the corresponding vertex labels of the above sequence.

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The degree factorial matrix of the $P_2 \times P_n$ is $DFM(P_2 \times P_n) =$

$$\left[\begin{array}{c|c} (2!)I_{4\times 4} + J_{4\times 4} - I_{4\times 4} & J_{4\times(2n-4)} \\ \hline J_{(2n-4)\times 4} & (3!)I_{(2n-4)\times(2n-4)} + J_{(2n-4)\times(2n-4)} - I_{(2n-4)\times(2n-4)} \end{array}\right]$$

The characteristic equation of degree factorial matrix of $P_2 \times P_n$ is

$$\begin{vmatrix} 2!-\lambda & 1 & 1 & 1 & 1 & . & . & 1 \\ 1 & 2!-\lambda & 1 & 1 & 1 & . & . & 1 \\ 1 & 1 & 2!-\lambda & 1 & 1 & . & . & 1 \\ 1 & 1 & 1 & 2!-\lambda & 1 & . & . & 1 \\ 1 & 1 & 1 & 1 & 3!-\lambda & . & . & 1 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & 1 & . & . & 3!-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)^3(\lambda-5)^{2n-5}\left(\lambda^2 - 2\lambda(n+3) + (2n+21)\right) = 0$$

The degree factorial spectrum of the grid graph is 3 times $1, (2n-5)$ times 5, $\left(n+3 \pm \sqrt{n^2+4n-12}\right)$.

Now the degree factorial energy of the grid graph $P_2 \times P_n$ is

$$DFE(P_2 \times P_n) = 1+1+1+5(2n-5)+2(n+3)$$
$$= 12n - 16 = 4(3n-4).$$

Theorem 2.5. The degree factorial energy of wheel graph W_n with n vertices, $n \geq 4$ is $(n-1)! + 6n - 6$.

Proof:

Let $W_n, n \geq 4$ be a wheel graph with n vertices. The degree sequence of wheel graph W_n is $\left((n-1), 3, 3, \dots\dots, 3\right)$. Let $v_1, v_2, \dots\dots, v_n$ be the corresponding vertex labels of the above sequence. The degree factorial matrix of wheel graph W_n is

$$DFM(W_n) = \left[\begin{array}{c|c} (n-1)! & J_{1\times(n-1)} \\ \hline J_{(n-1)\times 1} & (3!)I_{(n-1)\times(n-1)} + J_{(n-1)\times(n-1)} - I_{(n-1)\times(n-1)} \end{array}\right]$$

Now the characteristic equation of $DFM(W_n)$ is

$$\begin{vmatrix} (n-1)!-\lambda & 1 & . & . & . & 1 \\ 1 & 3!-\lambda & 1 & . & . & 1 \\ 1 & 1 & 3!-\lambda & . & . & 1 \\ . & . & . & . & . & . \\ 1 & 1 & . & . & . & 3!-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-5)^{n-2}(\lambda^2 - ((n-1)! + (n+4))\lambda + 2(n!) - (n-4)(n-1)! - (n-1)) = 0$$

The degree factorial spectrum of W_n is

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$(n-2)$ times 5, $\frac{(n+4)+(n-1)!\pm\sqrt{(n-1)![(n-1)!-8]+n^2+12n+12-2(n!)}}{2}$.

The degree factorial energy of W_n is

$$5+5+\cdots+5\ (n-2)\ times\ +\ \left|\frac{(n+4)+(n-1)!+\sqrt{(n-1)![(n-1)!-8+n^2+12n+12-2(n!)]}}{2}\right|$$

$$+\left|\frac{(n+4)+(n-1)!-\sqrt{(n-1)![(n-1)!-8+n^2+12n+12-2(n!)]}}{2}\right|$$

$$=5(n-2)+\frac{n+4+(n-1)!}{2}+\frac{n+4+(n-1)!}{2}$$

$$=5(n-2)+n+4+(n-1)!$$

$$=(n-1)!+6n-6.$$

Theorem 2.6. The degree factorial energy of star graph $S_n, n \geq 4$ is $(n-1)!+n-1$.

Proof:

Let $S_n, n \geq 4$ be the star graph with n vertices. The degree sequence of star graph S_n is $\big((n-1),1,1,\dots,1\big)$. Let $v_1, v_2, \dots\dots, v_n$ be the corresponding vertex labels of the above sequence. The degree factorial matrix of star graph $S_n, n \geq 4$ is

$$DFM(S_n)=\left[\begin{array}{c|c}(n-1)! & J_{1\times(n-1)} \\ \hline J_{(n-1)\times 1} & J_{(n-1)\times(n-1)}\end{array}\right]$$

The characteristic polynomial of degree factorial matrix for $S_n, n \geq 4$ is

$$\begin{vmatrix}(n-1)!-\lambda & 1 & 1 & . & . & . & 1 \\ 1 & 1!-\lambda & 1 & . & . & . & 1 \\ 1 & 1 & 1!-\lambda & . & . & . & 1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & 1 & 1 & . & . & . & 1!-\lambda\end{vmatrix}=0$$

$$\Rightarrow \lambda^{n-2}\big(\lambda^2-[(n-1)!+(n-1)]\lambda+(n!-[(n-1)!+n-1])\big)=0$$

$\therefore$ The degree factorial energy of star energy of star graph with $S_n, n \geq 4$ is $(n-1)!+n-1$.

Theorem 2.7. The degree factorial energy of the triangular snake TS_n is $n+24\left[\frac{n}{2}\right]-21$, where $m \geq 2$ and $n=2m+1$. ([n] denotes integral part of n).

Proof:

Let TS_n be a triangular snake graph on n vertices where $n=2m+1, m \geq 2$. Let $v_1, v_2, \dots, v_n$ be the corresponding vertex labels of the degree sequence of the triangular snake graph.

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The degree factorial matrix of TS_n is $DFM(TS_n) =$

$$\left[\begin{array}{c|c} (4!-1)I_{\left[\frac{n}{2}\right]-1\times\left[\frac{n}{2}\right]-1} + J_{\left[\frac{n}{2}\right]-1\times\left[\frac{n}{2}\right]-1} & J_{\left[\frac{n}{2}\right]-1\times\left[\frac{n}{2}\right]-1} \\ \hline J_{n-\left[\frac{n}{2}\right]+1\times n-\left[\frac{n}{2}\right]+1} & (2!-1)I_{n-\left[\frac{n}{2}\right]+1\times n-\left[\frac{n}{2}\right]+1} + J_{n-\left[\frac{n}{2}\right]+1\times n-\left[\frac{n}{2}\right]+1} \end{array}\right]$$

Now the characteristic equation of $DFM(TS_n)$ is

$$\left|\begin{array}{c|c} (4!-1)I_{\left[\frac{n}{2}\right]-1\times\left[\frac{n}{2}\right]-1} + J_{\left[\frac{n}{2}\right]-1\times\left[\frac{n}{2}\right]-1} & J_{\left[\frac{n}{2}\right]-1\times\left[\frac{n}{2}\right]-1} \\ \hline J_{n-\left[\frac{n}{2}\right]+1\times n-\left[\frac{n}{2}\right]+1} & (2!-1)I_{n-\left[\frac{n}{2}\right]+1\times n-\left[\frac{n}{2}\right]+1} + J_{n-\left[\frac{n}{2}\right]+1\times n-\left[\frac{n}{2}\right]+1} \end{array}\right| = 0$$

$$\Rightarrow (\lambda-1)^{\left[\frac{n}{2}\right]+1}(\lambda-23)^{\left[\frac{n}{2}\right]-2}\left(\lambda^2-(n+4!)\lambda+68+4!\left[\frac{n}{2}\right]\right)=0$$

The degree factorial spectrum of TS_n are $\left(\left[\frac{n}{2}\right]+1\right)$ times 1, $\left(\left[\frac{n}{2}\right]-2\right)$ times 23, $\frac{n+4!\pm\sqrt{n^2+48n-304+96\left[\frac{n}{2}\right]}}{2}$.

ow the degree factorial energy of the triangular snake graph TS_n is

$$1+1+\cdots.+1\,(\left[\frac{n}{2}\right]+1\ times)\ +23+23+\cdots+23(\left[\frac{n}{2}\right]-2\ times) + \left|\frac{n+4!+\sqrt{n^2+48n-304+96\left[\frac{n}{2}\right]}}{2}\right|$$

$$=\left[\frac{n}{2}\right]+1+23\left(\left[\frac{n}{2}\right]-2\right)+\frac{n+4!}{2}+\frac{n+4!}{2}=n+24\left[\frac{n}{2}\right]-21$$

∴ The degree factorial energy of TS_n is $n+24\left[\frac{n}{2}\right]-21$.

3.CONCLUSION

In this paper the new energy of a graph called degree factorial energy of a graph is introduced and the degree factorial spectrum and degree factorial energy of path graph, cycle graph, complete graph, grid graph, wheel graph, star graph and triangular snake graph were obtained.

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