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RESULTS ON COUPLED FIXED POINT WITH MIXED MONOTONE MAPPING IN VECTOR VALUED RECTANGULAR METRIC SPACES

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Abstract

In this manuscript, our primary focus is to establish some coupled fixed point results with mixed monotone mappings fulfilling a generalized contractive condition in vector valued rectangular metric spaces. The derived results extend and generalize various acknowledged results in the literature. An example is provided to illustrate our work.

Keywords: Riesz Space Valued Rectangular Metric Space, Mixed Monotone (MM) Property, Coupled Fixed Point.

1. INTRODUCTION

A fixed point of a function is a point in the domain of the function that maps to itself under the function. In the past several years, fixed point theory has become widely recognized as a potent and essential tool in the exploration of nonlinear analysis. Stefa Banach [3] proved a fundamental result in the study of metric space called Banach fixed point theorem in 1922, which state that "Every contraction in a complete metric space have a unique fixed point". Subsequently, the Banach fixed point theorem has been generalized by several authors in various metric spaces (see [2, 6, 8, 9, 12, 14]). A contraction mapping in Banach fixed point theorem is necessarily continuous. So, it is quite natural to ask, "Is there a contraction mapping that does not require to be continuous". Kannan [15] responded positively to this question and presented a novel contraction mapping called Kannan type mapping. Kannan type mapping may be discontinuous but for such mappings, he introduced a result that have a unique fixed point. After Kannan [15], numerous Mathematicians like Reich [16] and Chatterjea [10] presented the diversity of contraction conditions. In 2000, Branciari [6] presented the notation of rectangular metric space and proved some results related to fixed point. Cevik and Altun [8] presented the notation of vector metric space, where the metric is Riesz space valued. By merging rectangular metric space and vector metric space, we define new space called vector valued rectangular metric space. Consider a set E, a pair of elements $(\varsigma, \vartheta) \in E \times E$ is known as coupled fixed point (denoted as cfp) of the mapping $F : E \times E \to E$ if $\varsigma = F(\varsigma, \vartheta)$ and $\vartheta = F(\vartheta, \varsigma)$. The concept of mixed monotone mapping was first given by Bhaskar and Lakshmikantan [4], and discussed few results on cfp in ordered metric spaces. Afterwards, many scholars investigated results on cfp for mixed monotone mapping in various general metric spaces, for further information, we recommend the reader to [4, 5, 11, 13, 18, 19, 20]. In this work we establish few results related to cfp by using mixed monotone mappings in vector

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valued rectangular metric space. We extend and generalize the work of Ding et. al. and Bhaskar et. al. [4, 11]. Furthermore, we provide an example to clarify our work.

2. PRELIMINARIES

Here, we provide various definitions, results and notations that will be utilized later. For preliminaries and basic concept of Riesz space, one may refer, Aliprantis and Border [1].

Definition 2.1. [1] A partially ordered set (poset) is a pair $(E, \preceq)$ where E is any set and $\preceq$ is any binary relation which satisfy the following conditions:

(1) $\varsigma \preceq \varsigma$ (Reflexive)

(2) If $\varsigma \preceq \vartheta$ and $\vartheta \preceq \varsigma$ then $\varsigma = \vartheta$ (Antisymmetric)

(3) If $\varsigma \preceq \vartheta$ and $\vartheta \preceq \eta$ then $\varsigma \preceq \eta$ (Transitive)

for all $\varsigma, \vartheta, \eta \in E$.

Remark 2.2. [1] Let Q be a real vector space. Then poset $(Q, \preceq)$ is called partially ordered vector space if it satisfy the followings:

(1) $\varsigma \preceq \vartheta$ implies $\varsigma + \eta \preceq \vartheta + \eta$

(2) $\varsigma \preceq \vartheta$ implies $\lambda\varsigma \preceq \lambda\vartheta$, $\forall\, \varsigma, \vartheta, \eta \in Q$ and $\lambda > 0$.

Definition 2.3. [1] A lattice is a poset, where every pair of elements has a least upper bound and greatest lower bound. A Riesz space Q is a partially ordered vector space which is also a lattice under its ordering.

Here onwards Q shall denote a Riesz space.

Example 2.4. [1] The vector space $C_b(Q)$ of all bounded continuous real functions on the topological space under pointwise ordering defined by $f \preceq g$ whenever $f(\varsigma) \preceq g(\varsigma)$ for each $\varsigma \in Q$.

Notation: [1] The notation $\varsigma_m \downarrow \varsigma$ signifies that $\{\varsigma_m\}$ is a decreasing sequence in Q and infimum of ς_m is ς.

Definition 2.5. [1] A space Q is called Q-Archimedean if $\frac{1}{m}\varsigma \downarrow 0$, for each $m \in N$ and $\varsigma \in Q^+$ where $Q^+ = \{\varsigma \in Q : \varsigma \geq 0\}$.

Example 2.6. [1] The vector space $C(0,1)$ of all continuous functions on the open interval $(0,1)$ is Archimedean Riesz space under the usual pointwise ordering.

Lemma 2.7. [2] Consider a space Q and $\vartheta \preceq a\vartheta$, $\forall\, \vartheta \in Q_+$ also $0 \leq a < 1$, then $\vartheta = 0$.

Definition 2.8. [2] Let E be any non-empty set, then the mapping $\kappa : E \times E \to Q$ is vector metric if it satisfies the followings:

(a) $\kappa(\vartheta, \omega) = 0 \Leftrightarrow \vartheta = \omega$

(b) $\kappa(\vartheta, \omega) = \kappa(\omega, \vartheta)$

(c) $\kappa(\vartheta, \omega) \preceq \kappa(\vartheta, \eta) + \kappa(\eta, \omega)$, $\forall\, \eta, \vartheta, \omega \in E$

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Then (E, κ, Q) is called vector metric space (denoted as VMS).

Example 2.9. [2] A vector metric $\kappa : Q \times Q \to Q$ defined by:

$$\kappa(\varsigma, \vartheta) = |\varsigma - \vartheta|$$

is called absolute valued metric on Q.

Definition 2.10. [7] Let (E_1, κ_1, Q) and (E_2, κ_2, Q) be two VMS. A mapping $F: (E_1, \kappa_1, Q) \to (E_2, \kappa_2, Q)$ is vectorial continuous at ς_* if $\varsigma_n \xrightarrow{\kappa_1, Q} \varsigma^*$ in E_1 implies $F(\varsigma_n) \xrightarrow{\kappa_2, Q} F(\varsigma^*)$ in E_2. And, vectorial continuity of F at each elements of E implies F is vectorial continuous on E.

Definition 2.11. [2] A sequence $\{\vartheta_m\}$ in VMS (E, κ, Q) is called vectorial convergent (Q-convergent)

to some $\vartheta \in E$, denoted as $\vartheta_m \xrightarrow{\kappa, Q} \vartheta$ if $\exists$ r_m in Q s.t. $r_m \downarrow 0$ and $\kappa(\vartheta_m, \vartheta) \preceq r_m$. A sequence $\{\vartheta_m\}$ is Q-Cauchy, if $\exists\, r_m$ in Q s.t. $r_m \downarrow 0$ and $\forall\, m, p$, we have $\kappa(\vartheta_m, \vartheta_{m+p}) \preceq r_m$. If every Q-Cauchy sequence in E is Q-convergent to a limit in E, this implies that a VMS E is Q-complete.

Definition 2.12. [6] Let E be any non-empty set and the mapping $\kappa : E \times E \to R$ s.t. $\forall\, \vartheta, \omega \in E$ and for all distinct $\eta, \varsigma \in E$ s.t. $\eta, \varsigma \notin \{\vartheta, \omega\}$ is called rectangular metric if it satisfies the followings:

(a) $\kappa(\vartheta, \omega) = 0 \Leftrightarrow \vartheta = \omega$

(b) $\kappa(\vartheta, \omega) = \kappa(\omega, \vartheta)$

(c) $\kappa(\vartheta, \omega) \leq \kappa(\vartheta, \eta) + \kappa(\eta, \varsigma) + \kappa(\varsigma, \omega)$.

Then (E, κ) is called rectangular metric space (RMS).

Example 2.13. [6] Let $P = \{n : n \text{ is even and } 0 \leq n \leq 2\}$, $K = \left\{\frac{1}{n} : n \in N\right\}, E = P \cup K$. Define $\kappa : E \times E \to R^+$ as follows:

$$\kappa(\vartheta, \omega) = \begin{cases} 0 & \text{if } \vartheta = \omega \\ 1 & \text{if } \{\vartheta, \omega\} \subset P \text{ or } \{\vartheta, \omega\} \subset P \text{ and } \vartheta \neq \omega, \\ \vartheta & \text{if } \vartheta \in K, \omega \in P \\ \omega & \text{if } \vartheta \in P, \omega \in K. \end{cases}$$

Clearly, (E, κ) is RMS. But (E, κ) is not a metric space since $1 = \kappa(0, 2) > \kappa(0, \frac{1}{n}) + \kappa(\frac{1}{n}, 2)$, where $n \in N - \{1, 2\}$.

Definition 2.14. [14] Let E be any non-empty set, then the mapping $\kappa : E \times E \to Q$ is vector valued rectangular metric s.t. $\forall\, \vartheta, \omega \in E$ and for all distinct $\eta, \varsigma \in E$ s.t. $\eta, \varsigma \notin \{\vartheta, \omega\}$ if it satisfies the followings:

(a) $\kappa(\vartheta, \omega) = 0 \Leftrightarrow \vartheta = \omega$ (b) $\kappa(\vartheta, \omega) = \kappa(\omega, \vartheta)$

(b) $\kappa(\vartheta, \omega) = \kappa(\omega, \vartheta)$

(c) $\kappa(\vartheta, \omega) \preceq \kappa(\vartheta, \eta) + \kappa(\eta, \varsigma) + \kappa(\varsigma, \omega)$.

And the triplet (E, κ, Q) is vector valued rectangular metric space (VVRMS).

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Next, we provide an example of VVRMS that is not a VMS.

Example 2.15. [14] Let $Q = R^2, E = \{1,2,3,4\}$ and define $\kappa : E \times E \to Q$ s.t. $\kappa(\vartheta,\omega) = \kappa(\omega,\vartheta)$ and

$$\begin{cases} \kappa(\vartheta,\vartheta) = (0,0) \ \ \forall \vartheta \in E \\ \kappa(2,1) = \kappa(3,1) = \kappa(4,1) = \kappa(4,2) = \kappa(4,3) = (1,1) \\ \kappa(3,2) = (3,3) \end{cases}$$

Definition 2.16. [4] Let $(E,\preceq)$ be a poset. The mapping $F : E \times E \to E$ is said to has the mixed monotone (denoted as MM) property if for some $\varsigma,\vartheta \in E$,

$$\varsigma_1,\varsigma_2 \in G,\ \varsigma_1 \preceq \varsigma_2 \Rightarrow F(\varsigma_1,\vartheta) \preceq F(\varsigma_2,\vartheta)$$

and

$$\vartheta_1,\vartheta_2 \in E,\ \vartheta_1 \preceq \vartheta_2 \Rightarrow F(\varsigma,\vartheta_2) \preceq F(\varsigma,\vartheta_1).$$

3. MAIN RESULTS

The following results generalize the results of Ding et. al. [11] for VVRMS.

Lemma 3.1. Let $(E,\preceq_E)$ be a poset and (E,κ,Q) be a complete VVRMS with Q-Archimedean. Let the mapping $F : E \times E \to E$ be vectorial continuous. Further, the sequences $\{\varsigma_m\}$, $\{\vartheta_m\}$ in E defined by $\varsigma_m = F(\varsigma_{m-1},\vartheta_{m-1})$, $\vartheta_m = F(\vartheta_{m-1},\varsigma_{m-1})$ are Q-cauchy. Then there exist $\varsigma_*,\vartheta_* \in E$ such that $\varsigma_* = F(\varsigma_*,\vartheta_*)$ and $\vartheta_* = F(\vartheta_*,\varsigma_*)$.

Proof. We know that, $\{\varsigma_m\}$ and $\{\vartheta_m\}$ are Cauchy sequences, so there exists sequences r_m and j_m in Q with $r_m \downarrow 0$ and $j_m \downarrow 0$ such that $\kappa(\varsigma_m,\varsigma_{m+p}) \preceq r_m$ and $\kappa(\vartheta_m,\vartheta_{m+p}) \preceq j_m, \forall m$ and p. Since the space (E,κ,Q) is Q-complete, there exist $\varsigma_*,\vartheta_* \in E$ such that $\varsigma_m \xrightarrow{\kappa,Q} \varsigma_*$ and $\vartheta_m \xrightarrow{\kappa,Q} \vartheta_*$. Also, $(\varsigma_m,\vartheta_m) \xrightarrow{\kappa,Q} (\varsigma_*,\vartheta_*)$. And since F is vectorial continuous then we have $F(\varsigma_m,\vartheta_m) \xrightarrow{\kappa,Q} (\varsigma_*,\vartheta_*)$. So, there exist sequences a_m, b_m and c_m in Q with $\{a_m,b_m,c_m\} \downarrow 0$ s.t. $\kappa(\varsigma_m,\varsigma) \preceq_Q a_m$, $\kappa(\vartheta_m,\vartheta_*) \preceq_Q b_m$ and $\kappa\big(F(\varsigma_m,\vartheta_m),F(\varsigma_*,\vartheta_*)\big) \preceq_Q c_m$. Finally, we will claim that $F(\varsigma_*,\vartheta_*) = \varsigma_*$ and $F(\vartheta_*,\varsigma_*) = \vartheta_*$. Now

$$\begin{aligned} \kappa(F(\varsigma_*,\vartheta_*),\varsigma_*) &\preceq_Q \kappa(F(\varsigma_*,\vartheta_*),F(\varsigma_m,\vartheta_m)) + \kappa\big(F(\varsigma_m,\vartheta_m),F(\varsigma_{m+1},\vartheta_{m+1})\big) \\ &\quad + \kappa(F(\varsigma_{m+1},\vartheta_{m+1}),\varsigma_*) \\ &\preceq_Q c_m + \kappa(\varsigma_{m+1},\varsigma_{m+2}) + \kappa(\varsigma_{m+2},\varsigma_*) \\ &\preceq_Q c_m + r_m + a_m \ (\because a_{m+2} \preceq_Q a_m \text{ and } r_{m+1} \preceq_Q r_m) \\ &= z_m \end{aligned}$$

where $c_m + r_m + a_m = z_m$ and $z_m \downarrow 0$. We get $\kappa(F(\varsigma_*,\vartheta_*),\varsigma_*) = 0$, implies $F(\varsigma_*,\vartheta_*) = \varsigma_*$. Similarly, we can show $F(\vartheta_*,\varsigma_*) = \vartheta_*$.

Theorem 3.2. Let $(E,\preceq_E)$ be a poset and (E,κ,Q) be a VVRMS with Q-Archimedean. Let the mapping $F : E \times E \to E$ be vectorial continuous with MM property on E. Consider the following: (G1) there exists $a_1,a_2,a_3 \geq 0$ with $2a_1 + 3a_2 + 9a_3 < 2$ s.t.

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$$\kappa\big(F(\varsigma,\vartheta),F(s,t)\big) \preceq_Q \frac{a_1(\kappa(\varsigma,s)+\kappa(\vartheta,t))}{2} + \frac{a_2\big(\kappa(\varsigma,F(\varsigma,\vartheta))+\kappa(s,F(s,t))+\kappa(\vartheta,t)\big)}{2} + \frac{a_3\big(\kappa(\vartheta,F(s,t))+\kappa(t,F(\varsigma,\vartheta))+\kappa(\vartheta,t)\big)}{2}$$

for all $s \preceq_E \varsigma$ and $\vartheta \preceq_E t$.

(G2) for all $m = 0,1,2,3,\cdots$, $\kappa(\vartheta_m,\varsigma_m) \preceq_Q \kappa(\varsigma_{m+1},\varsigma_m) + \kappa(\vartheta_{m+1},\vartheta_m)$.

(G3) there exist $\varsigma_0,\vartheta_0 \in E$ such that $\varsigma_0 \preceq_E F(\varsigma_0,\vartheta_0)$ and $F(\vartheta_0,\varsigma_0) \preceq_E \vartheta_0$.

Then F has a cfp.

Proof. Let $\varsigma_m = F(\varsigma_{m-1},\vartheta_{m-1})$, $\vartheta_m = F(\vartheta_{m-1}, \varsigma_{m-1})$, where $m = 1,2,\cdots$. Since $\varsigma_0 \preceq_E F(\varsigma_0,\vartheta_0) = \varsigma_1$ and $\vartheta_1 = F(\vartheta_0,\varsigma_0) \preceq_E \vartheta_0$, then we have

$\varsigma_2 = F(\varsigma_1,\vartheta_1) = F(F(\varsigma_0,\vartheta_0), F(\vartheta_0,\varsigma_0))$

and

$\vartheta_2 = F(\vartheta_1,\varsigma_1) = F(F(\vartheta_0,\varsigma_0),F(\varsigma_0,\vartheta_0))$.

By using MM property on E, we have

$F(\varsigma_0,\vartheta) \preceq_E F(\varsigma_1,\vartheta)$, for any $\vartheta \in E$ and $F(\varsigma,\vartheta_0) \preceq_E F(\varsigma,\vartheta_1)$, for any $\varsigma \in E$.

Take $\vartheta = \vartheta_0$ and $\varsigma = \varsigma_1$, we get

$F(\varsigma_0,\vartheta_0) \preceq_E F(\varsigma_1,\vartheta_0)$ and $F(\varsigma_1,\vartheta_0) \preceq_E F(\varsigma_1,\vartheta_1)$

By transitivity we have,

$$F(\varsigma_0,\vartheta_0) \preceq_E F(\varsigma_1,\vartheta_1)$$

Similarly

$$F(\vartheta_1,\varsigma_1) \preceq_E F(\vartheta_0,\varsigma_0).$$

Continue the above process, we can easily verify that

$$\varsigma_0 \preceq_E F(\varsigma_0,\vartheta_0) = \varsigma_1 \preceq_E F(\varsigma_1,\vartheta_1) = \varsigma_2 \preceq_E \cdots \preceq_E F(\varsigma_n,\vartheta_n) = \varsigma_{n+1} \preceq_E \cdots$$

and

$$\cdots \preceq_E \vartheta_{n+1} = F(\vartheta_n,\varsigma_n) \preceq_E \cdots \preceq_E \vartheta_2 = F(\vartheta_1,\varsigma_1) \preceq_E \vartheta_1 = F(\vartheta_0,\varsigma_0) \preceq_E \vartheta_0.$$

Now, let

$\omega = \frac{\kappa(\varsigma_1,\varsigma_0)+\kappa(\vartheta_1,\vartheta_0)}{2}$.

Then by given hypothesis, we have

$$\kappa(\varsigma_2,\varsigma_1) = \kappa(F(\varsigma_1,\vartheta_1),F(\varsigma_0,\vartheta_0))$$

$$\preceq_Q \frac{a_1\big(\kappa(\varsigma1,\varsigma0)+\kappa(\vartheta1,\vartheta0)\big)}{2} + \frac{a_2\big(\kappa(\varsigma1,F(\varsigma1,\vartheta1))+\kappa(\varsigma0,F(\varsigma0,\vartheta0))+\kappa(\vartheta1,\vartheta0)\big)}{2} + \frac{a_3\big(\kappa(\vartheta_1,F(\varsigma_0,\vartheta_0))+\kappa(\vartheta_0,F(\varsigma_1,\vartheta_1))+\kappa(\vartheta_1,\vartheta_0)\big)}{2}$$

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$$= a_1\omega + \frac{a_2}{2}[\kappa(\varsigma_1,\varsigma_2) + \kappa(\varsigma_0,\varsigma_1) + \kappa(\vartheta_1,\vartheta_0)]$$

$$+\frac{a_3}{2}[\kappa(\vartheta_1,\varsigma_1) + \kappa(\vartheta_0,\varsigma_2) + \kappa(\vartheta_1,\vartheta_0)]$$

$$\preceq_Q a_1\omega + a_2\omega + \frac{a_2}{2}\kappa(\varsigma_1,\varsigma_2) + \frac{a_3}{2}[\kappa(\vartheta_1,\varsigma_1) + \kappa(\vartheta_0,\varsigma_0) + \kappa(\varsigma_0,\varsigma_1)$$

$$+\kappa(\varsigma_1,\varsigma_2) + \kappa(\vartheta_1,\vartheta_0)]$$

$$= (a_1 + a_2 + a_3)\omega + a_2 + \frac{a_3}{2}\kappa(\varsigma_1,\varsigma_2) + \frac{a_3}{2}[\kappa(\vartheta_1,\varsigma_1) + \kappa(\vartheta_0,\varsigma_0)]$$

$$\frac{2-a_2-a_3}{2}\kappa(\varsigma_2,\varsigma_1) \preceq_Q (a_1 + a_2 + a_3)\omega + \frac{a_3}{2}[\kappa(\vartheta_1,\vartheta_0) + \kappa(\vartheta_0,\varsigma_0) + \kappa(\varsigma_0,\varsigma_1) + \kappa(\vartheta_0,\varsigma_0)]$$

$$= (a_1 + a_2 + 2a_3)\omega + 2a_3\frac{\kappa(\varsigma_0,\vartheta_0)}{2}$$

$$\preceq_Q (a_1 + a_2 + 2a_3)\omega + \frac{2a_3(\kappa(\varsigma_1,\varsigma_0) + \kappa(\vartheta_1,\vartheta_0))}{2} \quad \text{(using (G2))}$$

Thus, we obtain

$$\kappa(\varsigma_2,\varsigma_1) \preceq_Q \frac{2(a_1 + a_2 + 4a_3)}{2- a_2 - a_3}\omega = h\omega$$

where $h = \frac{2(a_1 + a_2 + 4a_3)}{2- a_2 - a_3} < 1$, since $2a_1 + 3a_2 + 9a_3 < 2$.

Also, one can get

$$\kappa(\vartheta_1,\vartheta_2) = \kappa(F(\vartheta_0,\varsigma_0), F(\vartheta_1,\varsigma_1))$$

$$\preceq_Q \frac{a_1(\kappa(\vartheta_0,\vartheta_1) + \kappa(\varsigma_0,\varsigma_1))}{2}$$

$$+\frac{a_2\left(\kappa(\vartheta_0, F(\vartheta_0,\varsigma_0)) + \kappa(\vartheta_1, F(\vartheta_1,\varsigma_1)) + \kappa(\varsigma_0,\varsigma_1)\right)}{2}$$

$$+\frac{a_3(\kappa(\varsigma_0,F(\vartheta_1,\varsigma_1)) + \kappa(\varsigma_1,F(\vartheta_0,\varsigma_0)) + \kappa(\varsigma_0,\varsigma_1))}{2}$$

$$= a_1\omega + a_2\frac{\kappa(\vartheta_0,\vartheta_1) + \kappa(\vartheta_1,\vartheta_2) + \kappa(\varsigma_0,\varsigma_1)}{2}$$

$$+\frac{a_3(\kappa(\varsigma_0,\vartheta_2) + \kappa(\varsigma_1\vartheta_1) + \kappa(\varsigma_0,\varsigma_1))}{2}$$

$$\preceq_Q a_1\omega + a_2\omega + \frac{a_2}{2}\kappa(\vartheta_1,\vartheta_2) + \frac{a_3}{2}[\kappa(\varsigma_0,\vartheta_0) + \kappa(\vartheta_0,\vartheta_1) + \kappa(\vartheta_1,\vartheta_2)$$

$$+\kappa(\varsigma_1,\vartheta_1) + \kappa(\varsigma_0,\varsigma_1)]$$

$$\frac{2-a_2-a_3}{2}\kappa(\vartheta_1,\vartheta_2) \preceq_Q (a_1 + a_2 + a_3)\omega + \frac{a_3}{2}[\kappa(\varsigma_1,\varsigma_0) + \kappa(\varsigma_0,\vartheta_1) + \kappa(\vartheta_0,\vartheta_1)$$

$$+\kappa(\vartheta_0,\varsigma_0)]$$

$$= (a_1 + a_2 + 2a_3)\omega + 2a_3\frac{\kappa(\varsigma_0,\vartheta_0)}{2}$$

$$\preceq_Q (a_1 + a_2 + 2a_3)\omega + \frac{2a_3(\kappa(\varsigma_1,\varsigma_0)+ \kappa(\vartheta_1,\vartheta_0))}{2}$$

which also gives

$$\kappa(\vartheta_1,\vartheta_2) \preceq_Q \frac{2(a_1 + a_2 + 4a_3)}{2 - a_2 - a_3}\,\omega = h\omega$$

Similarly, we can find that

$$\kappa(\varsigma_3,\varsigma_2) \preceq_Q \frac{2(a_1 + a_2 + 4a_3)}{2 - a_2 - a_3}\left(\frac{\kappa(\varsigma_2,\varsigma_1) + \kappa(\vartheta_2,\vartheta_1)}{2}\right)$$

$$\vdots$$

$$\kappa(\varsigma_{m+1},\varsigma_m) \preceq_Q h^m\omega$$

$$\vdots$$

Also, by induction we deduce that

$$\kappa(\vartheta_{m+1},\vartheta_m) \preceq_Q h^m\omega, \qquad m = 1,2,3,\cdots$$

By using rectangular inequality and (G2), we have

$$\begin{aligned}\kappa(\varsigma_m,\varsigma_{m+2}) &\preceq_Q \kappa(\varsigma_m,\varsigma_{m+1}) + \kappa(\varsigma_{m+1},\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\varsigma_{m+2})\\ &\preceq_Q h^m\omega + \kappa(\varsigma_{m+2},\varsigma_{m+1}) + \kappa(\vartheta_{m+2},\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_m) + \kappa(\vartheta_m,\varsigma_{m+1})\\ &\quad +\kappa(\varsigma_{m+1},\varsigma_{m+2})\\ &\preceq_Q h^m\omega + h^{m+1}\omega + h^{m+1}\omega + h^m\omega + \kappa(\vartheta_{m+1},\vartheta_{m+1}) + \kappa(\varsigma_m,\varsigma_{m+1}) + h^{m+1}\omega\\ &\preceq_Q (4h^m + 3h^{m+1})\omega\end{aligned}$$

similarly, we can find

$$\kappa(\vartheta_m,\vartheta_{m+2}) \preceq_Q (4h^m + 3h^{m+1})\omega.$$

Next, we claim that $\{\varsigma_m\}$ and $\{\vartheta_m\}$ are Q-Cauchy sequences. For $\{\vartheta_m\}$, we consider $\kappa(\vartheta_m,\vartheta_{m+p})$ in two cases.

Case 1. If p is odd say $2n + 1$, $n \in N \cup \{0\}$, then we have

$$\begin{aligned}\kappa(\vartheta_m,\vartheta_{m+p}) &\preceq_Q \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+p})\\ &\preceq_Q \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \kappa(\vartheta_{m+3},\vartheta_{m+4})\\ &\quad +\kappa(\vartheta_{m+4},\vartheta_{m+p})\\ &\preceq_Q \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \cdots + \kappa(\vartheta_{m+2n-1},\vartheta_{m+2n})\\ &\quad + \kappa(\vartheta_{m+2n},\vartheta_{m+p})\\ &\preceq_Q (h^m + h^{m+1} + \cdots + h^{m+2n})\omega\\ &\preceq_Q \left\{\frac{h^m}{1-h}\omega\right\} \downarrow 0.\end{aligned}$$

Case 2. If p is even say $2n$, $n \in N$, then we have

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$$\kappa(\vartheta_m, \vartheta_{m+p}) \preceq_Q \kappa(\vartheta_m, \vartheta_{m+2}) + \kappa(\vartheta_{m+2}, \vartheta_{m+3}) + \kappa(\vartheta_{m+3}, \vartheta_{m+p})$$
$$\preceq_Q \kappa(\vartheta_m, \vartheta_{m+2}) + \kappa(\vartheta_{m+2}, \vartheta_{m+3}) + \kappa(\vartheta_{m+3}, \vartheta_{m+4}) + \kappa(\vartheta_{m+4}, \vartheta_{m+5}) + \kappa(\vartheta_{m+5}, \vartheta_{m+p})$$
$$\preceq_Q \kappa(\vartheta_m, \vartheta_{m+2}) + \kappa(\vartheta_{m+2}, \vartheta_{m+3}) + \cdots + \kappa(\vartheta_{m+2n-1}, \vartheta_{m+2n})$$
$$\preceq_Q (4h^m + 3h^{m+1})\omega + (h^{m+2} + h^{m+3} + \cdots + h^{m+2n-1})\omega$$
$$\preceq_Q \left(4h^m + 3h^{m+1} + \frac{h^{m+2}}{1-h}\right)\omega$$
$$= \left\{\frac{h^m}{1-h}(4 - 7h - 2h^2)\omega\right\} \downarrow 0$$

Therefore $\{\vartheta_m\}$ is Q-Cauchy sequence. Likewise, it can be demonstrated that $\{\varsigma_m\}$ is also a Q-Cauchy sequence. Now, by Lemma 2.1., it follows that F possesses a cfp $(\varsigma_*, \vartheta_*)$.

Theorem 3.3. Assuming that all conditions specified in Theorem 2.1 are fulfilled except for the continuity of F. Furthermore, let's consider that E exhibits the following:

(a) if a non-decreasing sequence $\{\varsigma_m\}$ converges to ς in E, implies $\varsigma_m \preceq_E \varsigma, \forall m \in N$;

(b) if a non-increasing sequence $\{\vartheta_m\}$ converges to ϑ in E, implies $\vartheta \preceq_E \vartheta_m, \forall m \in N$.

Then F has a cfp.

Proof. Let $\{\varsigma_m\}, \{\vartheta_m\}, \varsigma_*, \vartheta_*$ be as in Theorem 2.1. Now, by using rectangular inequality and

(G2), we have

$$\kappa(\vartheta_*, \varsigma_m) \preceq_Q \kappa(\vartheta_*, \vartheta_{m-1}) + \kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\vartheta_m, \varsigma_m)$$
$$\preceq_Q \kappa(\vartheta_*, \vartheta_{m-1}) + \kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\varsigma_{m+1}, \varsigma_m) + \kappa(\vartheta_{m+1}, \vartheta_m) \qquad (1)$$

and

$$\kappa(\vartheta_{m-1}, F(\varsigma *, \vartheta *)) \preceq_Q \kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\vartheta_m, \varsigma_m) + \kappa(\varsigma_m, F(\varsigma_*, \vartheta_*))$$
$$\preceq_Q \kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\varsigma_{m+1}, \varsigma_m) + \kappa(\vartheta_{m+1}, \vartheta_m) + \kappa(\varsigma_m, F(\varsigma_*, \vartheta_*)). \qquad (2)$$

Adding (1) and (2), we get

$$\kappa(\vartheta_*, \varsigma_m) + \kappa(\vartheta_{m-1}, F(\varsigma_*, \vartheta_*)) \preceq_Q \kappa(\vartheta_*, \vartheta_{m-1}) + 2[\kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\varsigma_{m+1}, \varsigma_m) + \kappa(\vartheta_{m+1}, \vartheta_m)] + \kappa(\varsigma_m, F(\varsigma_*, \vartheta_*)). \qquad (3)$$

By the assumptions (a) and (b), $\varsigma_m \preceq_E \varsigma_*$ and $\vartheta_* \preceq_E \vartheta_m$ for all $m \in N$.

By using (G1) and (3), we obtain

$$\kappa(F(\varsigma_*, \vartheta_*), \varsigma_m) = \kappa(F(\varsigma_*, \vartheta_*), F(\varsigma_{m-1}, \vartheta_{m-1}))$$

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
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$$\preceq_Q \ a_1 \frac{\kappa(\varsigma_*,\varsigma_{m-1}) + \kappa(\vartheta_*,\vartheta_{m-1})}{2}$$
$$+a_2 \frac{\kappa\big(\varsigma_*, F(\varsigma_*,\vartheta_*)\big) + \kappa\big(\varsigma_{m-1}, F(\varsigma_{m-1},\vartheta_{m-1})\big) + \kappa(\vartheta_*,\vartheta_{m-1})}{2}$$
$$+a_3 \frac{\kappa\big(\vartheta_*, F(\varsigma_{m-1},\vartheta_{m-1})\big) + \kappa\big(\vartheta_{m-1}, F(\varsigma_*,\vartheta_*)\big) + \kappa(\vartheta_*,\vartheta_{m-1})}{2}$$

$$= \frac{a_1}{2}\,\kappa(\varsigma_*,\varsigma_{m-1}) + \frac{a_1 + a_2 + a_3}{2}\,\kappa(\vartheta_*,\vartheta_{m-1})$$
$$+\frac{a_2}{2}\left[\kappa(\varsigma_{m-1},\varsigma_m) + \kappa\big(\varsigma_*, F(\varsigma_*,\vartheta_*)\big)\right] + \frac{a_3}{2}\left[\kappa(\vartheta_*,\varsigma_m) + \kappa(\vartheta_{m-1}, F(\varsigma_*,\vartheta_*))\right]$$

$$\preceq_Q \frac{a_1}{2}\,\kappa(\varsigma_*,\varsigma_{m-1}) + \frac{a_1 + a_2 + a_3}{2}\,\kappa(\vartheta_*,\vartheta_{m-1}) + \frac{a_2}{2}\,[\kappa(\varsigma_{m-1},\varsigma_m)$$
$$+\kappa(\varsigma_*, F(\varsigma_*,\vartheta_*))] \qquad + \frac{a_3}{2}(\kappa(\vartheta_*,\vartheta_{m-1}) + 2[\kappa(\vartheta_{m-1},\vartheta_m) + \kappa(\varsigma_{m+1},\varsigma_m)$$
$$+\kappa(\vartheta_{m+1},\vartheta_m)] + \kappa(\varsigma_m, F(\varsigma_*,\vartheta_*))) \text{ (using (3))}$$

$$\frac{2 - a_3}{2}\,\kappa(F(\varsigma_*,\vartheta_*),\varsigma_m) \preceq_Q \frac{a_1}{2}\,\kappa(\varsigma_*,\varsigma_{m-1}) + \frac{a_1 + a_2 + 2a_3}{2}\,\kappa(\vartheta_*,\vartheta_{m-1})$$
$$+\frac{a_2}{2}\,[\kappa(\varsigma_{m-1},\varsigma_m) + \kappa(\varsigma_*, F(\varsigma_*,\vartheta_*))]$$
$$+a_3[\kappa(\vartheta_{m-1},\vartheta_m) + \kappa(\varsigma_{m+1},\varsigma_m) + \kappa(\vartheta_{m+1},\vartheta_m)]$$

Then, it follows that

$$\kappa(F(\varsigma_*,\vartheta_*),\varsigma_m) \preceq_Q \frac{a_1}{2 - a_3}\kappa(\varsigma_*,\varsigma_{m-1}) + \frac{a_1 + a_2 + a_3}{2 - a_3}\kappa(\vartheta_*,\vartheta_{m-1}) + \frac{a_2}{2 - a_3}\,[\kappa(\varsigma_{m-1},\varsigma_m)$$
$$+\kappa(\varsigma_*, F(\varsigma_*,\vartheta_*))] \qquad + \frac{2a_3}{2 - a_3}[\kappa(\vartheta_{m-1},\vartheta_m) + \kappa(\varsigma_{m+1},\varsigma_m) + \kappa(\vartheta_{m+1},\vartheta_m)]. \quad (4)$$

Now, we will claim that $F(\varsigma_*,\vartheta_*) = \varsigma_*$ and $F(\vartheta_*,\varsigma_*) = \vartheta_*$.

$$\kappa\big(\varsigma_*, F(\varsigma_*,\vartheta_*)\big) \preceq_Q \kappa(\varsigma_*,\varsigma_{m-1}) + \kappa(\varsigma_{m-1},\varsigma_m) + \kappa(\varsigma_m, F(\varsigma_*,\vartheta_*))$$
$$\preceq_Q \kappa(\varsigma_*,\varsigma_{m-1}) + \kappa(\varsigma_{m-1},\varsigma_m) + \frac{a_1}{2 - a3}\,\kappa(\varsigma_*,\varsigma_{m-1}) + \frac{a_1 + a_2 + a_3}{2 - a_3}\,\kappa(\vartheta_*,\vartheta_{m-1})$$
$$+\frac{a_2}{2 - a_3}\left[\kappa(\varsigma_{m-1},\varsigma_m) + \kappa\big(\varsigma_*, F(\varsigma_*,\vartheta_*)\big)\right] + \frac{2a_3}{2 - a_3}\,[\kappa(\vartheta_{m-1},\vartheta_m)$$
$$+\kappa(\varsigma_{m+1},\varsigma_m) + \kappa(\vartheta_{m+1},\vartheta_m)]$$

$$\frac{2 - a_3 - a_2}{2 - a_3}\kappa\big(\varsigma, F(\varsigma_*,\vartheta_*)\big) \preceq_Q \frac{2 - a_3 + a_1}{2 - a_3}\kappa(\varsigma_*,\varsigma_{m-1}) + \frac{2 + a_2 - a_3}{2 - a_3}\kappa(\varsigma_{m-1},\varsigma_m) + \frac{a_2 + a_1 + a_3}{2 - a_3}\,\kappa(\vartheta_*,\vartheta_{m-1})$$

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
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$$+\frac{2a_3}{2-a_3}[\kappa(\vartheta_{m-1},\vartheta_m)+\kappa(\varsigma_{m+1},\varsigma_m)+\kappa(\vartheta_{m+1},\vartheta_m)]$$

$$\kappa\big(\varsigma_*,F(\varsigma_*,\vartheta_*)\big)\preceq_Q \frac{2+a_1-a_3}{2-a_2-a_3}\,a_m+\frac{2-a_2-a_3}{2-a_3-a_2}\,g_m+\frac{a_3+a_2+a_1}{2-a_2-a_3}\,b_m$$

$$+\frac{2a_3}{2-a_2-a_3}\,[h_m+g_m+h_m]$$

$$=\frac{a_1-a_3+2}{2-a_2-a_3}\,a_m+\frac{a_3-a_2+2}{2-a_2-a_3}\,g_m+\frac{a_3+a_2+a_1}{2-a_2-a_3}\,b_m$$

$$+\frac{4a_3}{2-a_2-a_3}\,h_m$$

since $\left\{\frac{2-a_3+a_1}{2-a_3-a_2},\frac{(2+a_3-a_2)}{2-a_3-a_2},\frac{a_1+a_2+a_3}{2-a_3-a_2},\frac{4a_3}{2-a_3-a_2}\right\}\geq 0$ and $\{a_m,g_m,b_m,h_m\}\downarrow 0$. Then we get

$\kappa(\varsigma_*,F(\varsigma_*,\vartheta_*))=0$. Hence $F(\varsigma_*,\vartheta_*)=\varsigma_*$, similarly we can find $F(\vartheta_*,\varsigma_*)=\vartheta_*$.

Theorem 3.4. Assuming that all the conditions specified in Theorem 2.1 (or Theorem 2.2) are fulfilled. Furthermore, if ς_0,ϑ_0 are comparable, and $2a_1+3a_2+3a_3<2$, then $\varsigma_*=\vartheta_*$.

Proof. W.L.O.G., we may let $\varsigma_0\preceq_E\vartheta_0$. Since F has MM property, implise $\varsigma_m\preceq_E\vartheta_m,\forall\, m\in N$. Then, we have

$$\kappa(\vartheta_m,\varsigma_m)=\kappa(F(\vartheta_{m-1},\varsigma_{m-1}),F(\varsigma_{m-1},\vartheta_{m-1}))$$

$$\preceq_Q \frac{a_1(\kappa(\vartheta_{m-1},\varsigma_{m-1})+\kappa(\varsigma_{m-1},\vartheta_{m-1}))}{2}$$

$$+\frac{a_2\big(\kappa(\vartheta_{m-1},F(\vartheta_{m-1},\varsigma_{m-1}))+\kappa(\varsigma_{m-1},F(\varsigma_{m-1},\vartheta_{m-1}))+\kappa(\varsigma_{m-1},\vartheta_{m-1})\big)}{2}$$

$$+\frac{a_3\Big(\kappa(\varsigma_{m-1},F(\varsigma_{m-1},\vartheta_{m-1}))+\kappa\big(\vartheta_{m-1},F(\vartheta_{m-1},\varsigma_{m-1})\big)+\kappa(\varsigma_{m-1},\vartheta_{m-1})\Big)}{2}$$

$$\kappa(\vartheta_m,\varsigma_m)\preceq_Q a_1\kappa(\vartheta_{m-1},\varsigma_{m-1})+a_2(\kappa(\vartheta_{m-1},\vartheta_m)+\kappa(\varsigma_{m-1},\varsigma_m)+\frac{\kappa(\varsigma_{m-1},\vartheta_{m-1})}{2}$$

$$+\frac{a_3(\kappa(\varsigma_{m-1},\varsigma_m)+\kappa(\vartheta_{m-1},\vartheta_m)+\kappa(\varsigma_{m-1},\vartheta_{m-1}))}{2}$$

$$=(a_1+\frac{a_2+a_3}{2})\kappa(\varsigma_{m-1},\vartheta_{m-1})+\frac{a_2+a_3}{2}[\kappa(\vartheta_{m-1},\vartheta_m)+\kappa(\varsigma_{m-1},\varsigma_m)]$$

$$\preceq_Q (a_1+\frac{a_2+a_3}{2})[\kappa(\varsigma_{m-1},\varsigma_*)+\kappa(\varsigma_*,\vartheta_*)+\kappa(\vartheta_*,\vartheta_{m-1})]$$

$$+\frac{a_2+a_3}{2}[\kappa(\varsigma_{m-1},\varsigma_*)+\kappa(\varsigma_*,\vartheta_*)+\kappa(\vartheta_*,\varsigma_m)+\kappa(\vartheta_{m-1},\vartheta_*)+\kappa(\vartheta_*,\varsigma_*)$$
$$+\kappa(\varsigma_*,\vartheta_m)]$$

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
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$$= \left(a_1 + \frac{3(a_2 + a_3)}{2}\right)\kappa(\varsigma_*,\vartheta_*) + (a_1 + a_2 + a_3)[\kappa(\vartheta_{m-1},\vartheta_*) + \kappa(\varsigma_{m-1},\varsigma_*)]$$
$$+ \frac{a_2 + a_3}{2}[\kappa(\varsigma_m,\varsigma_*) + +\kappa(\vartheta_m,\varsigma_*)] \quad (5)$$

Also by rectangular inequality, we can write

$$\kappa(\varsigma_*,\vartheta_*) \preceq_Q \kappa(\varsigma_*,\varsigma_m) + \kappa(\varsigma_m,\vartheta_m) + \kappa(\vartheta_m,\vartheta_*)$$
$$\preceq_Q \kappa(\varsigma_*,\varsigma_m) + \left(a_1 + \frac{3(a_2+a_3)}{2}\right)\kappa(\varsigma_*,\vartheta_*) + (a_1 + a_2 + a_3)$$
$$[\kappa(\vartheta_{m-1},\vartheta_*) + \kappa(\varsigma_{m-1},\varsigma_*)] + \frac{a_2 + a_3}{2}[\kappa(\varsigma_m,\varsigma_*) + \kappa(\vartheta_m,\varsigma_*)]$$
$$+\kappa(\vartheta_m,\vartheta_*) \text{ (by using (5))}$$

$$\left(1 - a_1 - \frac{3(a_2+a_3)}{2}\right)\kappa(\varsigma *,\vartheta *) \preceq_Q (a_1 + a_2 + a_3)[\kappa(\vartheta_{m-1},\vartheta_*) + \kappa(\varsigma_{m-1},\varsigma_*)]$$
$$+ \left(1 + \frac{(a_2+a_3)}{2}\right)[\kappa(\varsigma_m,\varsigma_*) + \kappa(\vartheta_m,\varsigma_*)]$$
(6)

On the other hand, since $\varsigma_m \xrightarrow{\kappa,Q} \varsigma_*$ and $\vartheta_m \xrightarrow{\kappa,Q} \vartheta_*$, then ∃ $\{p_m\}$ and $\{g_m\}$ in Q with $p_m \downarrow 0$ and

$g_m \downarrow 0$ such that

$$\kappa(\varsigma_m,\varsigma_*) \preceq_Q p_m,\ \kappa(\vartheta_m,\vartheta_*) \preceq_Q g_m. \quad (7)$$

Then from (6) and (7), we have

$$\left(\frac{2 - 2a_1 - 3a_2 - 3a_3}{2}\right)\kappa(\varsigma_*,\vartheta_*) \preceq_Q (a_2 + a_3 + a_1)(p_{m-1} + g_{m-1}) + \left(\frac{a_2 + a_3 + 2}{2}\right)(p_m + g_m)$$

$$\preceq_Q \frac{2 + 2a_1 + 3a_2 + 3a_3}{2}(p_{m-1} + g_{m-1})(\because\ p_m \preceq_Q p_{m-1} \text{ and } g_m \preceq_Q g_{m-1})$$

$$\kappa(\varsigma_*,\vartheta_*) \preceq_Q \frac{2 + 2a_1 + 3a_2 + 3a_3}{2 - 2a_1 - 3a_2 - 3a_3}(p_{m-1} + g_{m-1}).$$

Since $2a_1 + 3a_2 + 3a_3 < 2$, implies $\kappa(\varsigma_*,\vartheta_*) = 0$. Hence $\varsigma_* = \vartheta_*$.

Theorem 3.5. Let $(E,\preceq_E)$ be a poset and (E,κ,Q) be a complete VVRMS with Q-Archimedean. Let the mapping $F : E \times E \to E$ has MM property. Consider the following:

(M1) there exists $\gamma \in \left[0,\frac{2}{3}\right)$ such that ∀ $s \preceq_E \varsigma$ and $\vartheta \preceq_E t$,

$$\kappa\big(F(\varsigma,\vartheta),F(s,t)\big) \preceq_Q \gamma K_F(\varsigma,\vartheta,s,t),$$

where $K_F(\varsigma,\vartheta,s,t) \in \left\{\frac{\kappa(\varsigma,s)+\kappa(\vartheta,t)}{2}, \frac{\kappa(\varsigma,F(\varsigma,\vartheta))+\kappa(s,F(s,t))+\kappa(\vartheta,t)}{2}\right\}$.

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
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(M2) there exist $\varsigma_0, \vartheta_0 \in E$ such that $\varsigma_0 \preceq_E F(\varsigma_0, \vartheta_0)$ and $F(\vartheta_0, \varsigma_0) \preceq_E \vartheta_0$.

(M3) F is vectorial continuous or E satisfies the following:

(i) if a non-decreasing sequence $\{\varsigma_m\}$ converges to ς in E, then $\varsigma_m \preceq_E \varsigma, \forall\, m \in N$;

(ii) if a non-increasing sequence $\{\vartheta_m\}$ converges to ϑ in E, then $\vartheta \preceq_E \vartheta_m, \forall\, m \in N$.

Then there exists $\varsigma_*, \vartheta_* \in E$ such that $\varsigma_* = F(\varsigma_*, \vartheta_*)$ and $\vartheta_* = F(\vartheta_*, \varsigma_*)$. Further, if ς_0, ϑ_0 are comparable, then $\varsigma_* = \vartheta_*$.

Proof. Let $\varsigma_m = F(\varsigma_{m-1}, \vartheta_{m-1}),\ \vartheta_m = F(\vartheta_{m-1}, \varsigma_{m-1})$, where $m = 1, 2, \cdots$

Since F has the MM property on E and by (M2), we get

$$\varsigma_0 \preceq_E F(\varsigma_0, \vartheta_0) = \varsigma_1 \preceq_E F(\varsigma_1, \vartheta_1) = \varsigma_2 \preceq_E \cdots \preceq_E F(\varsigma_m, \vartheta_m) = \varsigma_{m+1} \preceq_E \cdots$$

and

$$\cdots \preceq_E \vartheta_{m+1} = F(\vartheta_m, \varsigma_m) \preceq_E \cdots \preceq_E \vartheta_2 = F(\vartheta_1, \varsigma_1) \preceq_E \vartheta_1 = F(\vartheta_0, \varsigma_0) \preceq_E \vartheta_0.$$

By using (M1), we have

$$\begin{aligned}\kappa(\varsigma_{m+1}, \varsigma_m) &= \kappa(F(\varsigma_m, \vartheta_m), F(\varsigma_{m-1}, \vartheta_{m-1})) \\ &\preceq_Q \gamma K_F(\varsigma_m, \vartheta_m, \varsigma_{m-1}, \vartheta_{m-1})\end{aligned}$$

where

$$K_F(\varsigma_m, \vartheta_m, \varsigma_{m-1}, \vartheta_{m-1}) \in \{\frac{\kappa(\varsigma_m, \varsigma_{m-1}) + \kappa(\vartheta_m, \vartheta_{m-1})}{2}, \frac{\kappa(\varsigma_m, F(\varsigma_m, \vartheta_m)) + \kappa(\varsigma_{m-1}, F(\varsigma_{m-1}, \vartheta_{m-1})) + \kappa(\vartheta_m, \vartheta_{m-1})}{2}\}$$

$$= \left\{\frac{\kappa(\varsigma_m, \varsigma_{m-1}) + \kappa(\vartheta_m, \vartheta_{m-1})}{2}, \frac{\kappa(\varsigma_m, \varsigma_{m+1}) + \kappa(\varsigma_m, \varsigma_{m-1}) + \kappa(\vartheta_m, \vartheta_{m-1})}{2}\right\}.$$

Now, we consider two cases:

1. If $K_F(\varsigma_m, \vartheta_m, \varsigma_{m-1}, \vartheta_{m-1}) = \frac{\kappa(\varsigma_m, \varsigma_{m-1}) + \kappa(\vartheta_m, \vartheta_{m-1})}{2}$, then

$$\begin{aligned}\kappa(\varsigma_{m+1}, \varsigma_m) &\preceq_Q \frac{\gamma\left(\kappa(\varsigma_m, \varsigma_{m-1}) + \kappa(\vartheta_m, \vartheta_{m-1})\right)}{2} \\ &\preceq_Q \frac{2\gamma}{2-\gamma}\left(\frac{\kappa(\varsigma_m, \varsigma_{m-1}) + \kappa(\vartheta_m, \vartheta_{m-1})}{2}\right),\end{aligned}$$

2. If $K_F(\varsigma_m, \vartheta_m, \varsigma_{m-1}, \vartheta_{m-1}) = \frac{\kappa(\varsigma_m, \varsigma_{m+1}) + \kappa(\varsigma_m, \varsigma_{m-1}) + \kappa(\vartheta_m, \vartheta_{m-1})}{2}$, then

$$\kappa(\varsigma_{m+1}, \varsigma_m) \preceq_Q \frac{\gamma}{2}\kappa(\varsigma_{m+1}, \varsigma_m) + \frac{\gamma\left(\kappa(\varsigma_m, \varsigma_{m-1}) + \kappa(\vartheta_m, \vartheta_{m-1})\right)}{2}$$

$$\left(\frac{2-\gamma}{2}\right)\kappa(\varsigma_{m+1}, \varsigma_m) \preceq_Q \frac{\gamma\left(\kappa(\varsigma_m, \varsigma_{m-1}) + \kappa(\vartheta_m, \vartheta_{m-1})\right)}{2}.$$

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
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Thus, we have

$$\kappa(\varsigma_{m+1},\varsigma_m) \preceq_Q \frac{2\gamma}{2-\gamma}\left(\frac{\kappa(\varsigma_m,\varsigma_{m-1}) + \kappa(\vartheta_m,\vartheta_{m-1})}{2}\right), \forall m \in N.$$

By a similar proof, one can also show that

$$\kappa(\vartheta_{m+1},\vartheta_m) \preceq_Q \frac{2\gamma}{2-\gamma}\left(\frac{\kappa(\varsigma_m,\varsigma_{m-1}) + \kappa(\vartheta_m,\vartheta_{m-1})}{2}\right), \forall m \in N.$$

Let $h = \frac{2\gamma}{2-\gamma}$ and $= \frac{\kappa(\varsigma_1,\varsigma_0)+\kappa(\vartheta_1,\vartheta_0)}{2}$. It follows from that $\gamma \in \left[0,\frac{2}{3}\right)$ that $h \in [0,1)$. Then for all

$m \in N$, we conclude that

$$\kappa(\varsigma_{m+1},\varsigma_m) \preceq_Q h^m\omega, \tag{8}$$

and

$$\kappa(\vartheta_{m+1},\vartheta_m) \preceq_Q h^m\omega. \tag{9}$$

Now from rectangular inequality and (8), we have

$$\begin{aligned}\kappa(\varsigma_{m+1},\varsigma_{m-1}) &\preceq_Q \kappa(\varsigma_{m+1},\varsigma_{m+2}) + \kappa(\varsigma_{m+2},\varsigma_m) + \kappa(\varsigma_m,\varsigma_{m-1})\\ &\preceq_Q (h^{m+1} + h^{m-1})\omega + \kappa(\varsigma_{m+2},\varsigma_m)\end{aligned} \tag{10}$$

Similarly from rectangular inequality and (9), one can find

$$\kappa(\vartheta_{m+1},\vartheta_{m-1}) \preceq_Q (h^{m+1} + h^{m-1})\omega + \kappa(\vartheta_{m+2},\vartheta_m). \tag{11}$$

By using (M1), we have

$$\kappa(\varsigma_{m+2},\varsigma_m) = \kappa\big(F(\varsigma_{m+1},\vartheta_{m+1}),F(\varsigma_{m-1},\vartheta_{m-1})\big) \preceq_Q \gamma K_F(\varsigma_{m+1},\vartheta_{m+1},\varsigma_{m-1},\vartheta_{m-1})$$

where

$$K_F(\varsigma_{m+1},\vartheta_{m+1},\varsigma_{m-1},\vartheta_{m-1}) \in \{\frac{\kappa(\varsigma_{m+1},\varsigma_{m-1})+\kappa(\vartheta_{m+1},\vartheta_{m-1})}{2}, \frac{\kappa(\varsigma_{m+1},F(\varsigma_{m+1},\vartheta_{m+1})) + \kappa(\varsigma_{m-1},F(\varsigma_{m-1},\vartheta_{m-1})) + \kappa(\vartheta_{m+1},\vartheta_{m-1})}{2}\}$$

$$= \left\{\begin{array}{c}\dfrac{\kappa(\varsigma_{m+1},\varsigma_{m-1}) + \kappa(\vartheta_{m+1},\vartheta_{m-1})}{2},\\ \dfrac{\kappa(\varsigma_{m+1},\varsigma_{m+2}) + \kappa(\varsigma_{m-1},\varsigma_m) + \kappa(\vartheta_{m+1},\vartheta_{m-1})}{2}\end{array}\right\}.$$

Now, we consider two cases:

1. If $K_F(\varsigma_{m+1},\vartheta_{m+1},\varsigma_{m-1},\vartheta_{m-1}) = \frac{\kappa(\varsigma_{m+1},\varsigma_{m-1}) + \kappa(\vartheta_{m+1},\vartheta_{m-1})}{2}$, then

$$\begin{aligned}\kappa(\varsigma_{m+2},\varsigma_m) &\preceq_Q \frac{\gamma\big(\kappa(\varsigma_{m+1},\varsigma_{m-1})+ \kappa(\vartheta_{m+1},\vartheta_{m-1})\big)}{2}\\ &\preceq_Q \frac{\gamma}{2}\left[\kappa(\varsigma_{m+1},\varsigma_{m+2}) + \kappa(\varsigma_{m+2},\varsigma_m) + \kappa(\varsigma_m,\varsigma_{m-1}) + \kappa(\vartheta_{m+1},\vartheta_{m-1})\right]\end{aligned}$$

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
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$\frac{2-\gamma}{2}\kappa(\varsigma_{m+2},\varsigma_m) \preceq_Q \frac{\gamma}{2}\,[(h^{m+1}+h^{m-1})\omega+(h^{m+1}+h^{m-1})\omega \ +\kappa(\vartheta_{m+2},\vartheta_m)]$ (using(8) and (11))

$\kappa(\varsigma_{m+2},\varsigma_m) \preceq_Q \frac{\gamma}{2-\gamma}\,[(h^{m+1}+h^{m-1})2\omega\ + \kappa(\vartheta_{m+2},\vartheta_m)].$ (12)

2. If $K_F(\varsigma_{m+1},\vartheta_{m+1},\varsigma_{m-1},\vartheta_{m-1}) \ = \frac{\kappa(\varsigma_{m+1},\varsigma_{m+2})\,+\,\kappa(\varsigma_{m-1},\varsigma_m)+\,\kappa(\vartheta_{m+1},\vartheta_{m-1})}{2}$, then

$$\kappa(\varsigma_{m+2},\varsigma_m) \ \preceq_Q \gamma\frac{\kappa(\varsigma_{m+1},\varsigma_{m+2})\,+\,\kappa(\varsigma_{m-1},\varsigma_m)+\,\kappa(\vartheta_{m+1},\vartheta_{m-1})}{2}$$

$$\preceq_Q \frac{\gamma}{2}\,[(h^{m+1}+h^{m-1})\omega+\kappa(\vartheta_{m+1},\vartheta_{m-1})] \text{ (using(8))}$$

$$\preceq_Q \frac{\gamma}{2}\,[(h^{m+1}+h^{m-1})\omega\ +\ (h^{m+1}+h^{m+1})\omega\ +\ \kappa(\vartheta_{m+2},\vartheta_m)] \text{ (using(11))}$$

$$\preceq_Q \ \frac{\gamma}{2-\gamma}\,[(h^{m+1}+h^{m-1})2\omega\ +\ \kappa(\vartheta_{m+2},\vartheta_m)]\ \left(\because \frac{\gamma}{2} \leq \frac{\gamma}{2-\gamma}\right)$$

Thus, we have

$$\kappa(\varsigma_{m+2},\varsigma_m) \preceq_Q \frac{\gamma}{2-\gamma}[(h^{m+1}\ +h^{m-1})2\omega\ +\ \kappa(\vartheta_{m+2},\vartheta_m)]. \tag{13}$$

Similarly, we can obtain

$$\kappa(\vartheta_{m+2},\vartheta_m) \preceq_Q \frac{\gamma}{2-\gamma}\ [(h^{m+1}\ +h^{m-1})2\omega\ +\ \kappa(\varsigma_{m+2},\varsigma_m). \tag{14}$$

By using (13) and (14), we have

$$\kappa(\varsigma_{m+2},\varsigma_m) \ \preceq_Q \ \frac{\gamma}{2-\gamma}\left[(h^{m+1}\ +h^{m-1})2\omega\ +\ \frac{\gamma}{2-\gamma}\ [(h^{m+1}\ +h^{m-1})2\omega\ + \kappa(\varsigma_{m+2},\varsigma_m)]\right]$$

$$= \left[\frac{\gamma}{2-\gamma}+\ \left(\frac{\gamma}{2-\gamma}\right)^2\right](h^{m+1}\ +h^{m-1})2\omega\ +\ \left(\frac{\gamma}{2-\gamma}\right)^2\kappa(\varsigma_{m+2},\varsigma_m)$$

$$\left(1-\left(\frac{\gamma}{2-\gamma}\right)^2\right)\kappa(\varsigma_{m+2},\varsigma_m) \ \preceq_Q \ \frac{\gamma(2\ -\ \gamma)+\gamma^2}{(2-\gamma)^2}\ \ h^{m-1}[1\ +\ h^2\ ]2\omega$$

$$\frac{4(1\ -\ \gamma)}{(2\ -\ \gamma)^2}\kappa(\varsigma_{m+2},\varsigma_m) \preceq_Q \frac{4\gamma}{(2\ -\ \gamma)^2}\ h^{m-1}\ [1\ +\ h^2\ ]\omega$$

$$\kappa(\varsigma_{m+2},\varsigma_m) \preceq_Q \frac{\gamma}{1-\gamma}\ h^{m-1}\ [1\ +\ h^2\ ]\omega, \tag{15}$$

similarly

$$\kappa(\vartheta_{m+2},\vartheta_m) \preceq_Q \frac{\gamma}{1-\gamma}\ h^{m-1}\ [1\ +\ h^2\ ]\omega. \tag{16}$$

Next, we claim that $\{\varsigma_m\}$ and $\{\vartheta_m\}$ are Q-Cauchy sequences. For $\{\vartheta_m\}$, we consider $\kappa(\vartheta_m,\vartheta_{m+p})$ in two cases.

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
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Case 1. If p is odd say $2n + 1, n \in N \cup \{0\}$, then we have

$$\begin{aligned}\kappa(\vartheta_m,\vartheta_{m+p}) &\preceq_Q \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+p})\\
&\preceq_Q \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \kappa(\vartheta_{m+3},\vartheta_{m+4})\\
&\quad +\kappa(\vartheta_{m+4},\vartheta_{m+p})\\
&\preceq_Q \kappa(\vartheta_m,\vartheta_{m+1}) + \kappa(\vartheta_{m+1},\vartheta_{m+2}) + \cdots + \kappa(\vartheta_{m+2n-1},\vartheta_{m+2n})\\
&\quad + \kappa(\vartheta_{m+2n},\vartheta_{m+p})\\
&\preceq_Q (h^m + h^{m+1} + \cdots + h^{m+2n})\omega\\
&\preceq_Q \left\{\frac{h^m}{1-h}\omega\right\} \downarrow 0.\end{aligned}$$

Case 2. If p is even say $2n, n \in N$, then we have

$$\begin{aligned}\kappa(\vartheta_m,\vartheta_{m+p}) &\preceq_Q \kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \kappa(\vartheta_{m+3},\vartheta_{m+p})\\
&\preceq_Q \kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \kappa(\vartheta_{m+3},\vartheta_{m+4}) + \kappa(\vartheta_{m+4},\vartheta_{m+5})\\
&\quad +\kappa(\vartheta_{m+5},\vartheta_{m+p})\\
&\preceq_Q \kappa(\vartheta_m,\vartheta_{m+2}) + \kappa(\vartheta_{m+2},\vartheta_{m+3}) + \cdots + \kappa(\vartheta_{m+2n-1},\vartheta_{m+2n})\\
&\preceq_Q \frac{\gamma}{1-\gamma}h^{m-1}[1+h^2]\omega + (h^{m+2} + h^{m+3} + \cdots + h^{m+2n-1})\omega \text{ (using (15))}\\
&\preceq_Q \left[\frac{\gamma}{1-\gamma}h^{m-1}[1+h^2] + \frac{h^{m+2}}{1-h}\right]\omega\\
&= \left\{\left[\frac{\gamma}{1-\gamma}[1+h^2] + \frac{h^3}{1-h}\right]h^{m-1}\omega\right\} \downarrow\end{aligned}$$

Therefore $\{\vartheta_m\}$ is Q-Cauchy sequence. Likewise, it can be demonstrated that $\{\varsigma_m\}$ is also a Q-Cauchy sequence. Suppose by hypothesis that E is Q-complete. So there exits ς_*, ϑ_* in E such that $\varsigma_m \xrightarrow{\kappa,Q} \varsigma_*$ and $\vartheta_m \xrightarrow{\kappa,Q} \vartheta_*$.

It remains to prove

$$\varsigma_* = F(\varsigma_*,\vartheta_*) \text{ and } \vartheta_* = F(\vartheta_*,\varsigma_*). \tag{17}$$

Now, if F is vectorial continuous, then by Lemma 2.1., (17) obviously holds. Now, suppose that (i) and (ii) of (M3) hold. Since $\varsigma_m \xrightarrow{\kappa,Q} \varsigma_*$ and $\vartheta_m \xrightarrow{\kappa,Q} \vartheta_*$, then there exists sequences $\{a_m\}$ and $\{b_m\}$ in Q s.t. $a_m \downarrow 0$, $b_m \downarrow 0$, and

$$\kappa(\varsigma_m,\varsigma_*) \preceq_Q a_m \text{ and } \kappa(\vartheta_m,\vartheta_*) \preceq_Q b_m. \tag{18}$$

We know that sequences $\{\varsigma_m\}$ and $\{\vartheta_m\}$ are Q-Cauchy. So there exists sequences $\{r_m\}$ and $\{j_m\}$ in Q s.t. $r_m \downarrow 0$, $j_m \downarrow 0$, and for all m and p

$$\kappa(\varsigma_m,\varsigma_{m+p}) \preceq_Q r_m,\ \kappa(\vartheta_m,\vartheta_{m+p}) \preceq_Q j_m. \tag{19}$$

Since $\varsigma_m \preceq_E \varsigma_*$ and $\vartheta_m \preceq_E \vartheta_*$ for all $m \in N$, then we have

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
Journal of Tianjin University Science and Technology
ISSN (Online):0493-2137
E-Publication: Online Open Access
Vol: 57 Issue: 05:2024
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$$\kappa(F(\varsigma_*,\vartheta_*),\varsigma_m) = \kappa\big(F(\varsigma_*,\vartheta_*),F(\varsigma_{m-1},\vartheta_{m-1})\big) \preceq_Q \gamma K_F(\varsigma_*,\vartheta_*,\varsigma_{m-1},\vartheta_{m-1}),$$

Where

$$K_F(\varsigma_*,\vartheta_*,\varsigma_{m-1},\vartheta_{m-1}) \in \left\{\frac{\kappa(\varsigma_*,\varsigma_{m-1}) + \kappa(\vartheta_*,\vartheta_{m-1})}{2}, \frac{\kappa(\varsigma_*,F(\varsigma_*,\vartheta_*)) + \kappa(\varsigma_{m-1},F(\varsigma_{m-1},\vartheta_{m-1})) + \kappa(\vartheta_*,\vartheta_{m-1})}{2}\right\}$$

$$= \left\{\frac{\kappa(\varsigma_*,\varsigma_{m-1}) + \kappa(\vartheta_*,\vartheta_{m-1})}{2}, \frac{\kappa(\varsigma_*,F(\varsigma_*,\vartheta_*)) + \kappa(\varsigma_{m-1},\varsigma_m) + \kappa(\vartheta_*,\vartheta_{m-1})}{2}\right\}.$$

Now, we consider two cases:

1. If $K_F(\varsigma_*,\vartheta_*,\varsigma_{m-1},\vartheta_{m-1}) = \frac{\kappa(\varsigma_*,\varsigma_{m-1}) + \kappa(\vartheta_*,\vartheta_{m-1})}{2}$, then

$$\kappa(F(\varsigma_*,\vartheta_*),\varsigma_m) \preceq_Q \gamma\left(\frac{\kappa(\varsigma_*,\varsigma_{m-1}) + \kappa(\vartheta_*,\vartheta_{m-1})}{2}\right)$$

$$\preceq_Q \frac{\gamma}{2}\left[\kappa(\varsigma_*,F(\varsigma_*,\vartheta_*)) + \kappa(F(\varsigma_*,\vartheta_*),\varsigma_m) + \kappa(\varsigma_m,\varsigma_{m-1}) + \kappa(\vartheta_*,\vartheta_{m-1})\right]$$

$$\frac{2-\gamma}{2}\kappa(F(\varsigma_*,\vartheta_*),\varsigma_m) \preceq_Q \frac{\gamma}{2}\left[\kappa(\varsigma_*,F(\varsigma_*,\vartheta_*)) + \kappa(\varsigma_m,\varsigma_{m-1}) + \kappa(\vartheta_*,\vartheta_{m-1})\right]$$

$$\kappa(F(\varsigma_*,\vartheta_*),\varsigma_m) \preceq_Q \frac{\gamma}{2-\gamma}\left[\kappa\big(\varsigma_*,F(\varsigma_*,\vartheta_*)\big) + \kappa(\varsigma_m,\varsigma_{m-1}) + \kappa(\vartheta_*,\vartheta_{m-1})\right].$$

2. If $K_F(\varsigma_*,\vartheta_*,\varsigma_{m-1},\vartheta_{m-1}) = \frac{\kappa(\varsigma_*,F(\varsigma_*,\vartheta_*))+\kappa(\varsigma_{m-1},\varsigma_m)+\kappa(\vartheta_*,\vartheta_{m-1})}{2}$, then

$$\kappa(F(\varsigma_*,\vartheta_*),\varsigma_m) \preceq_Q \frac{\gamma}{2}\left[\kappa(\varsigma_*,F(\varsigma_*,\vartheta_*)) + \kappa(\varsigma_{m-1},\varsigma_m) + \kappa(\vartheta_*,\vartheta_{m-1})\right]$$

$$\preceq_Q \frac{\gamma}{2-\gamma}\left[\kappa(\varsigma_*,F(\varsigma_*,\vartheta_*)) + \kappa(\varsigma_{m-1},\varsigma_m) + \kappa(\vartheta_*,\vartheta_{m-1})\right] \left(\because \frac{\gamma}{2} \leq \frac{\gamma}{2-\gamma}\right)$$

Thus, from both the cases, we conclude that

$$\kappa(F(\varsigma_*,\vartheta_*),\varsigma_m) \preceq_Q \frac{\gamma}{2-\gamma}\left[\kappa(\varsigma_*,F(\varsigma_*,\vartheta_*)) + \kappa(\varsigma_{m-1},\varsigma_m) + \kappa(\vartheta_*,\vartheta_{m-1})\right]. \quad (20)$$

Next, we claim that $(\varsigma_*,\vartheta_*)$ is cfp of F. By using (20), we have

$$\kappa(F(\varsigma_*,\vartheta_*),\varsigma_*) \preceq_Q \kappa(F(\varsigma_*,\vartheta_*),\varsigma_m) + \kappa(\varsigma_m,\varsigma_{m-1}) + \kappa(\varsigma_{m-1},\varsigma_*)$$

$$\preceq_Q \frac{\gamma}{2-\gamma}\left[\kappa(\varsigma_*,F(\varsigma_*,\vartheta_*)) + \kappa(\varsigma_{m-1},\varsigma_m) + \kappa(\vartheta_*,\vartheta_{m-1})\right]$$

$$+\kappa(\varsigma_m,\varsigma_{m-1}) + \kappa(\varsigma_{m-1},\varsigma_*)$$

$$\frac{2(1-\gamma)}{2-\gamma}\kappa(F(\varsigma_*,\vartheta_*),\varsigma_*) \preceq_Q \frac{2}{2-\gamma}\kappa(\varsigma_{m-1},\varsigma_m) + \frac{\gamma}{2-\gamma}\kappa(\vartheta_*,\vartheta_{m-1}) + \kappa(\varsigma_{m-1},\varsigma *)$$

$$\kappa(F(\varsigma_*,\vartheta_*),\varsigma_*) \preceq_Q \frac{1}{1-\gamma}\kappa(\varsigma_{m-1},\varsigma_m) + \frac{\gamma}{2(1-\gamma)}\kappa(\vartheta_*,\vartheta_{m-1})$$

$$+ \frac{2-\gamma}{2(1-\gamma)}\kappa(\varsigma_{m-1},\varsigma_*)$$

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
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Vol: 57 Issue: 05:2024
DOI: 10.5281/zenodo.11144179

$\preceq_Q \frac{1}{1-\gamma} r_{m-1} + \frac{\gamma}{2(1-\gamma)} b_{m-1} + \frac{2-\gamma}{2(1-\gamma)} a_{m-1}$ (by using (18) and (19)).

Since $\left\{\frac{1}{1-\gamma}, \frac{\gamma}{2(1-\gamma)}, \frac{2-\gamma}{2(1-\gamma)}\right\} \geq 0$ and $\{a_m, b_m, r_m\} \downarrow 0$. Then we get $\kappa(F(\varsigma_*, \vartheta_*), \varsigma_*) = 0$, implies $F(\varsigma_*, \vartheta_*) = \varsigma_*$. Similarly we can find $F(\vartheta_*, \varsigma_*) = \vartheta_*$.

Moreover, if ς_0, ϑ_0 are comparable. W.L.O.G., we may let $\varsigma_0 \preceq_E \vartheta_0$. Since F has a MM property, implies $\varsigma_m \preceq_E \vartheta_m \ \forall\, m \in N$. Thus, we have

$$\kappa(\varsigma_*, \vartheta_*) \preceq_Q \kappa(\varsigma_*, \varsigma_m) + \kappa(\varsigma_m, \vartheta_m) + \kappa(\vartheta_m, \vartheta_*)$$

$$\preceq_Q \kappa(F(\varsigma_{m-1}, \vartheta_{m-1}), F(\vartheta_{m-1}, \varsigma_{m-1})) + \kappa(\varsigma_*, \varsigma_m) + \kappa(\vartheta_m, \vartheta_*)$$

$$\preceq_Q \gamma K_F(\varsigma_{m-1}, \vartheta_{m-1}, \vartheta_{m-1}, \varsigma_{m-1}) + \kappa(\varsigma_*, \varsigma_m) + \kappa(\vartheta_m, \vartheta_*)$$

where

$$K_F(\varsigma_{m-1}, \vartheta_{m-1}, \vartheta_{m-1}, \varsigma_{m-1}) \in \left\{\frac{\kappa(\varsigma_{m-1}, \vartheta_{m-1}) + \kappa(\varsigma_{m-1}, \vartheta_{m-1})}{2},\right.$$

$$\left.\frac{\kappa(\varsigma_{m-1}, F(\varsigma_{m-1}, \vartheta_{m-1})) + \kappa(\vartheta_{m-1}, F(\vartheta_{m-1}, \varsigma_{m-1})) + \kappa(\varsigma_{m-1}, \vartheta_{m-1})}{2}\right\}$$

$$= \left\{\kappa(\varsigma_{m-1}, \vartheta_{m-1}), \frac{\kappa(\varsigma_{m-1}, \varsigma_m) + \kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\varsigma_{m-1}, \vartheta_{m-1})}{2}\right\}.$$

Here, we consider two cases:

1. If $K_F(\varsigma_{m-1}, \vartheta_{m-1}, \vartheta_{m-1}, \varsigma_{m-1}) = \kappa(\varsigma_{m-1}, \vartheta_{m-1})$, then we have

$$\kappa(\varsigma_*, \vartheta_*) \preceq_Q \gamma\kappa(\varsigma_{m-1}, \vartheta_{m-1}) + \kappa(\varsigma_*, \varsigma_m) + \kappa(\vartheta_m, \vartheta_*)$$

$$\preceq_Q \gamma[\kappa(\varsigma_{m-1}, \varsigma_*) + \kappa(\varsigma_*, \vartheta_*) + \kappa(\vartheta_*, \vartheta_{m-1})] + \kappa(\varsigma_*, \varsigma_m) + \kappa(\vartheta_m, \vartheta_*)$$

$$(1-\gamma)\kappa(\varsigma *, \vartheta *) \preceq_Q \gamma[a_{m-1} + b_{m-1}] + a_m + b_m$$

$$\kappa(\varsigma_*, \vartheta_*) \preceq_Q \frac{1+\gamma}{1-\gamma}[a_{m-1} + b_{m-1}] (\because a_m \preceq_Q a_{m-1} \text{ and } b_m \preceq_Q b_{m-1}).$$

2. If $K_F(\varsigma_{m-1}, \vartheta_{m-1}, \vartheta_{m-1}, \varsigma_{m-1}) = \frac{\kappa(\varsigma_{m-1}, \varsigma_m) + \kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\varsigma_{m-1}, \vartheta_{m-1})}{2}$, then

$$\kappa(\varsigma_*, \vartheta_*) \preceq_Q \frac{\gamma}{2}[\kappa(\varsigma_{m-1}, \varsigma_m) + \kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\varsigma_{m-1}, \vartheta_{m-1})] + \kappa(\varsigma_*, \varsigma_m) + \kappa(\vartheta_m, \vartheta_*)$$

$$\preceq_Q \frac{\gamma}{2}[\kappa(\varsigma_{m-1}, \varsigma_m) + \kappa(\vartheta_{m-1}, \vartheta_m) + \kappa(\varsigma_{m-1}, \varsigma_*) + \kappa(\varsigma_*, \vartheta_*) + \kappa(\vartheta_*, \vartheta_{m-1})] + \kappa(\varsigma_*, \varsigma_m) + \kappa(\vartheta_m, \vartheta_*)$$

$$(1-\gamma)\kappa(\varsigma_*, \vartheta_*) \preceq_Q \frac{\gamma}{2}[r_{m-1} + j_{m-1} + a_{m-1} + b_{m-1}] + a_m + b_m$$

$$\kappa(\varsigma_*, \vartheta_*) \preceq_Q \frac{2+\gamma}{2(1-\gamma)}[a_{m-1} + b_{m-1}] + \frac{\gamma}{2(1-\gamma)}[r_{m-1} + j_{m-1}].$$

Since $\left\{\frac{1+\gamma}{1-\gamma}, \frac{2+\gamma}{2(1-\gamma)}, \frac{\gamma}{2(1-\gamma)}\right\} \geq 0$ and $\{a_{m-1}, b_{m-1}, r_{m-1}, j_{m-1}\} \downarrow 0$, then from both the cases we conclude that $\kappa(\varsigma_*, \vartheta_*) = 0$. This means that $\varsigma_* = \vartheta_*$.

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
Journal of Tianjin University Science and Technology
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Corollary 3.6. Let $(E, \preceq_E)$ be a poset and (E, κ, Q) be a complete VVRMS with Q-Archimedean. Let the mapping $F : E \times E \to E$ has MM property. Consider the following:

(A1) there exists $\gamma \in [0, 1)$ such that $\forall\, s \preceq_E \varsigma, \vartheta \preceq_E t$ and

$$\kappa\big(F(\varsigma,\vartheta), F(s,t)\big) \preceq_Q \frac{\gamma}{2}[\kappa(\varsigma, s) + \kappa(\vartheta, t)],$$

(A2) there exist$\varsigma_0, \vartheta_0 \in E$ such that $\varsigma_0 \preceq_E F(\varsigma_0, \vartheta_0)$ and $F(\vartheta_0, \varsigma_0) \preceq_E \vartheta_0$.

(A3) F is vectorial continuous or E satisfies the following:

(i) if a non-decreasing sequence $\{\varsigma_m\}$ converges to ς in E, then $\varsigma_m \preceq_E \varsigma, \forall\, m \in N$;

(ii) if a non-increasing sequence $\{\vartheta_m\}$ converges to ϑ in E, then $\vartheta \preceq_E \vartheta_m, \forall\, m \in N$.

Then there exists $\varsigma_*, \vartheta_* \in E$ such that $\varsigma_* = F(\varsigma_*, \vartheta_*)$ and $\vartheta_* = F(\vartheta_*, \varsigma_*)$. Further, if ς_0, ϑ_0 are comparable, then $\varsigma_* = \vartheta_*$.

Proof. The conclusion can be drawn from Theorem 3.5.

Example 3.7. Let $E = R$ and $Q = R^2$ with coordinatwise ordering defined by $(\varsigma_1, \vartheta_1) \preceq (\varsigma_2, \vartheta_2)$ if and only if $\varsigma_1 \preceq \varsigma_2$ and $(\vartheta_1, \vartheta_2)$. Let $\kappa(\varsigma, \vartheta) = (|\varsigma - \vartheta|, r|\varsigma - \vartheta|)$ with $r > 0$ and define a function $F : E \times E \to E$ as $(\varsigma, \vartheta) = \frac{\varsigma - \vartheta}{12}$. Clearly (E, κ, Q) is complete VVRMS, since for all distinct $\varsigma, \vartheta, p, q \in E$, we have

$$\begin{aligned}
\kappa(\varsigma,\vartheta) &= (|\varsigma - \vartheta|, r|\varsigma - \vartheta|) \\
&= (|\varsigma - \vartheta + p - p + q - q|, r|\varsigma - \vartheta + p - p + q - q|) \\
&\preceq (|\varsigma - p| + |p - q| + |q - \vartheta|, r(|\varsigma - p| + |p - q| + |q - \vartheta|)) \\
&= (|\varsigma - p|, r|\varsigma - p|) + (|p - q|, r|p - q|) + (|q - \vartheta|, r|q - \vartheta|) \\
&= \kappa(\varsigma, p) + \kappa(p, q) + \kappa(q, \vartheta).
\end{aligned}$$

Also R^2 is Q-Archimedean with defined ordering. Now for all $\varsigma, \vartheta, s, t \in E$, we have

$$\begin{aligned}
\kappa\big(F(\varsigma,\vartheta), F(s,t)\big) &= \kappa\left(\frac{\varsigma - \vartheta}{12}, \frac{s - t}{12}\right) \\
&= \frac{1}{12}(|\varsigma - \vartheta - s + t|, r|\varsigma - \vartheta - s + t|) \\
&\preceq \frac{1}{12}(|\varsigma - s| + |\vartheta - t|, r(|\varsigma - s| + |\vartheta - t|)) \\
&= \frac{1}{12}[(|\varsigma - s|, r|\varsigma - s|) + (|\vartheta - t|, r|\vartheta - t|)] \\
&= \frac{1}{12}[\kappa(\varsigma, r) + \kappa(\vartheta, t)]
\end{aligned}$$

That is, $\kappa(F(\varsigma,\vartheta), F(s,t)) \preceq \gamma K_F(\varsigma, \vartheta, s, t)$, where $\gamma = \frac{1}{6} \in \left[0, \frac{2}{3}\right)$ and

$$K_F(\varsigma, \vartheta, s, t) \in \left\{\frac{\kappa(\varsigma, s) + \kappa(\vartheta, t)}{2}, \frac{\kappa\big(\varsigma, F(\varsigma,\vartheta)\big) + \kappa\big(s, F(s,t)\big) + \kappa(\vartheta, t)}{2}\right\}.$$

Also note that if $\varsigma_0 = -5, \vartheta_0 = 4$, then $\varsigma_0 \preceq F(\varsigma_0, \vartheta_0)$ and $F(\vartheta_0, \varsigma_0) \preceq \vartheta_0$. Therefore all the criteria specified in Theorem 3.5. have been fulfilled. Hence $(0, 0)$ is a cfp of F, which is unique.

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