Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
Journal of Tianjin University Science and Technology
ISSN (Online):0493-2137
E-Publication: Online Open Access
Vol: 56 Issue: 07: 2023
DOI10.17605/OSF.IO/HDJEA

ON APPROXIMATION OF COMPLEX CAUCHY PRINCIPAL VALUE INTEGRALS AND HYPER SINGULAR INTEGRALS

SWAGATIKA DAS*
Research Scholar, Biju Patanaik University and Technology, Chhend Colony, Rourkella, Odisha, India. *Corresponding Author Email: swagatikadp@gmail.com

GEETANJALI PRADHAN
Department of Mathematics & Humanities, Odisha University of Technology and Research Bhubaneswar, Odisha, India. Email: geetanjalipradhan65@gmail.com

RABINDRA NATH DAS
Department of Mathematics and Computer Science, Gangadhar Meher (Autonomous) College, Sambalpur, Odisha, India. Email: rabi_das08@gmail.com

Abstract

Some quadrature rules of a degree of precision six, eight, and ten have been formulated for numerical evaluation of complex Cauchy principal value integrals and their asymptotic errors have been obtained. The rules which have been constructed in this paper involve neither the derivative nor its approximation at any of the nodes on which the rules are based. Besides, a few more quadrature rules from the rules derived in the first instant have also been constructed following the technique of extrapolation and their asymptotic error estimates have also been obtained. Some standard test integrals of the Cauchy principal value type and Hyper singular type have been numerically evaluated by each of the rules constructed in this paper.

Keywords: Asymptotic error, Cauchy principal value, degree of precision, error bound, error constant, Hadamard finite part-integral (HFP), hyper singular.

2020 mathematical subject classification: No: 65D30, 65D32.

1. INTRODUCTION

A complex Cauchy principal value integral along a directed line segment L, joining points $z_0 - h \, and \, z_0 + h$ in the complex plane C is given by

$$I(f) = \int_{z_0-h}^{z_0+h} \frac{f(z)}{z-z_0} dz \qquad (1.1)$$

where f(z) is an analytic function in a simply connected domain Ω containing the line segment L and z_0 is the mid-point of the line segment L. Such integrals occur quite often in different branches of engineering, theory of elasticity, aerodynamics, scattering theory, etc. Most importantly, it occurs very often in contour integration, which in turn, becomes an essential tool in applied mathematics. Extensive research works have been conducted by many researchers such as Acharya and Das [1], Milovanovic, Acharya, and Pattanaik [12], Das and Pradhan ([4],[5],[6]) Das and Hota [7], Chawla [3], Longman [11], Hunter [9], Davis and Rabinowitz [8] and others. The main objective of this paper is to construct some interpolatory type of rules of a degree of precision six and eight and form these rules (of precision six and eight), some quadrature rules of precision eight and ten have been obtained by extrapolation. The rules have been derived in the forthcoming section.

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2. FORMULATIONS OF THE RULE

For the construction of the rule the following set of nodes is chosen:

$$z_0, z_0 \pm h, z_0 \pm ih,\ z_0 \pm \alpha h \quad \text{where } i = \sqrt{-1} \text{ and } 0 < \alpha < 1.$$

It may be noted that these nodes include the five nodes $z_0, z_0 \pm h, z_0 \pm ih$, which Birkhoff-Young [2] have used first for the formulation of an interpolatory type of rule of precision five for approximation of integrals of an analytic function on a line segment lying in the domain of analyticity.

Let the rule based on these nodes be denoted by $R(f;\alpha)$ and suppose

$$R(f;\alpha) = Af(z_0) + B[f(z_0 + h) - f(z_0 - h)]$$
$$+C[f(z_0 + ih) - f(z_0 - ih)] + D[f(z_0 + \alpha h) - f(z_0 - \alpha h)]\ . \quad (2.1)$$

The weights A, B, C, and D are to be determined so that;

$$I((z - z_0)^k) = R((z - z_0)^k);\ \text{for k= 0, 1, 3, 5.} \quad (2.2)$$

It is pertinent to note here that

$$I((z - z_0)^{2k}) = R((z - z_0)^{2k});\ \text{for k= 1,2,3, …}$$

i.e. it integrates all monomials of even degree since the nodes in the proposed rule given in (2.1) are symmetrically situated about the point $z = z_0$. Using the identities given in (2.2), the following set of four linear equations in the unknowns A, B, C, and D are

$$\left.\begin{aligned} A &= 0 \\ B + iC + \alpha D &= 1 \\ 3B - 3iC + 3\alpha^3 D &= 1 \\ 5B + 5iC + 5\alpha^5 D &= 1 \end{aligned}\right\};\quad i = \sqrt{-1}. \quad (2.3)$$

On solving the set of linear equations in the weights A, B, C, and D given in (2.3) we arrive at

$$A=0\ ,\ B = \frac{2(2-5\alpha^2)}{15(1-\alpha^2)}\ ,\ C = \frac{5\alpha^2-1}{15i(1+\alpha^2)}\ ,\ D = \frac{-4}{5\alpha(\alpha^4-1)}. \quad (2.4)$$

Thus, the rule $R(f;\alpha)$ given in equation (2.1) with weights A, B, C, and D as given in (2.4) represents a one-parameter (α) family of six point rules (since the weight A of f(z_0) is zero) integrating all polynomials of degree at least six.

From this one- parameter family of rules certain specific rules can be obtained for some suitable values of $'\alpha'$ with a smaller number of nodes.

Some specific rules

A set of two 4-point rules can be obtained from the rule (2.1) by choosing suitable values of the parameter $'\alpha'$ without altering the algebraic degree of precision of the rule (2.1) which is six.

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i) $\boldsymbol{\alpha} = \sqrt{\frac{2}{5}}$

For this value of α, we note that the weight B =0 and the corresponding rule which is denoted by $R_1(f)$ is given as

$$R_1(f) = \frac{-i}{21}[f(z_0 + ih) - f(z_0 - ih)] + \frac{10\sqrt{10}}{21}\left[f\left(z_0 + \sqrt{\frac{2}{5}}h\right) - f\left(z_0 - \sqrt{\frac{2}{5}}h\right)\right] \quad (2.5)$$

ii) $\boldsymbol{\alpha} = \frac{1}{\sqrt{5}}$

Again, for this choice of α, the weight C in (2.4) is found to be zero and the rule given in (2.1) now boils down to a 4- point rule which we denote by $R_2(f)$ and is given by

$$R_2(f) = \frac{1}{6}[f(z_0 + h) - f(z_0 - h)] + \frac{5\sqrt{5}}{6}\left[f\left(z_0 + \frac{1}{\sqrt{5}}h\right) - f\left(z_0 - \frac{1}{\sqrt{5}}h\right)\right]. \quad (2.6)$$

DEGREE OF PRECISION OF THE RULES $\boldsymbol{R_1(f)}$ AND $\boldsymbol{R_2(f)}$:

Denoting the truncation errors associated with the rules $R_1(f)$ and $R_2(f)$ by $E_1(f)$ and $E_2(f)$ respectively in the approximation of the integral given in (1.1) we have

$$\text{I (f)} = R_i(f) + E_i(f)\ ; \quad i = 1, 2.$$

and for i = 1, 2

$$E_i((z - z_0)^k) = 0; \quad k = 0(1)6$$

which implies that the degree of precision of both the rules $R_1(f)$ and $R_2(f)$ is at least six.

Further

$$E_i((z - z_0)^7) = \begin{cases} \frac{136}{525}h^7 \neq 0; & i = 1 \\ -\frac{32}{525}h^7 \neq 0; & i = 2 \end{cases} .$$

So, each of the rules $R_1(f)$ and $R_2(f)$ is a 4- point rule and the algebraic degree of precision of each is exactly six.

For the approximation of the integral (1.1) by the rules $R_1(f)$ and $R_2(f)$, we need to determine the values of the function at four points only, however for any other values of α in (0,1), one is to evaluate the function at six points, although all these rules are of precision six.

Two of the complex nodes viz $z_0 \pm ih$ associated with the rule $R_1(f)$ are off the contour of integration whereas all the nodes in case of the rule $R_2(f)$ are on the line of integration L. It is shown in Numerical Result section that the rule $R_2(f)$ numerically integrates more accurately than the rule $R_1(f)$ although both have the same degree of precision i.e six.

Thus, it appears that a rule having some or all of its nodes off the path of integration does not integrate as accurately as a rule having the same precision but all its nodes on the path of integration.

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iii) $\boldsymbol{\alpha} = \sqrt{\frac{5}{21}}$

It may be noted here that:

$$I((z-z_0)^k) - R((z-z_0)^k;\alpha) = 0; \text{ for k=0(1)6}$$

and

$$I((z-z_0)^7) - R((z-z_0)^7;\alpha) = \frac{4(42\alpha^2-10)}{105}h^7 \neq 0;$$

but it is zero if $\alpha = \sqrt{\frac{5}{21}}$. Hence for this value of α, the degree of precision of the rule (2.1) with weights:

$$A= 0,\ B=\frac{17}{120},\quad C=\frac{-2i}{195},\quad D=\frac{441}{520\times\sqrt{\frac{5}{21}}}$$

increases from six to eight.

We denote this rule of precision eight by $R_3(f)$ and it is given as

$$R_3(f)=\frac{17}{120}[f(z_0+h) - f(z_0-h)] - \frac{2i}{195}[f(z_0+ih) - f(z_0-ih)] + \frac{441}{520\times\sqrt{\frac{5}{21}}}\left[f\left(z_0+\sqrt{\frac{5}{21}}h\right) - f\left(z_0-\sqrt{\frac{5}{21}}h\right)\right] \quad . \quad (2.7)$$

A glance at the rules $R_1(f), R_2(f)$, and $R_3(f)$ reveal that the weights associated with the nodes which are off the contour are imaginary and those associated with the nodes on the contour are real.

The rules $R_1(f), R_2(f)$, and $R_3(f)$ may be termed as ***basic rules*****.**

Next, we construct two more quadrature rules; one is of precision eight and the other is of precision ten from the basic rules $R_1(f), R_2(f)$, and $R_3(f)$. These rules may be called as ***composite rules,*** being a linear combination of ***basic rules*** of the same precision.

iv) Construction of the composite rules:

The first *composite rule* to be formulated here by the method of extrapolation is of precision eight and it is different from the rule $R_3(f)$ given in (2.7). We denote this rule of precision eight by $R_{1,2}(f)$ and it is constructed by using the two *basic rules* $R_1(f)$ and $R_2(f)$ having the same degree of precision i.e. six.

The second *composite rule* is constructed by using the *basic rule* $R_3(f)$ and the *composite rule* $R_{1,2}(f)$ and we denote this rule by $R_{1,2,3}(f)$. It is pertinent to note here that both the rules $R_3(f)$ and $R_{1,2}(f)$ are of the same degree of precision.

Denoting the truncation errors by $E_1(f)$, $E_2(f)$, and $E_3(f)$ incurred in approximation of the integral (1.1), by the rules $R_1(f)$, $R_2(f)$, and $R_3(f)$ respectively we have

$$I(f) = R_1(f) + E_1(f)\ , \qquad (2.8)$$

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$$\text{I (f)} = R_2(f) + E_2(f), \tag{2.9}$$

and

$$\text{I (f)} = R_3(f) + E_3(f). \tag{2.10}$$

Assuming f(z) to be infinitely differentiable, the truncation errors associated with the rules under references can be expressed as

$$E_1(f) = \frac{136}{525}\frac{f^{(7)}(z_0)}{7!}h^7 + \frac{88}{1125}\frac{f^{(9)}(z_0)}{9!}h^9 + \cdots$$

$$E_2(f) = -\frac{32}{525}\frac{f^{(7)}(z_0)}{7!}h^7 - \frac{128}{1125}\frac{f^{(9)}(z_0)}{9!}h^9 - \cdots$$

and

$$E_3(f) = -\frac{64}{735}\frac{f^{(9)}(z_0)}{9!}h^9 - \frac{8384}{101871}\frac{f^{(11)}(z_0)}{11!}h^{11} - \cdots .$$

Now multiplying (2.8) by $\frac{4}{21}$ and (2.9) by $\frac{17}{21}$, then adding these results we obtain

$$\text{I (f)} = \left[\frac{4}{21}R_1(f) + \frac{17}{21}R_2(f)\right] + \left[\frac{4}{21}E_1(f) + \frac{17}{21}E_2(f)\right]$$

$$= R_{1,2}(f) + E_{1,2}(f) \tag{2.11}$$

where

$$R_{1,2}(f) = \frac{1}{21}\left(4R_1(f) + 17R_2(f)\right) \tag{2.12}$$

is the desired *composite rule* meant for the approximate evaluation of I(f) and the corresponding truncation error committed in this approximation is given by

$$E_{1,2}(f) = \frac{4}{21}E_1(f) + \frac{17}{21}E_2(f)$$

$$= -\frac{608}{7875}\frac{f^{(9)}(z_0)h^9}{9!} - \frac{10688}{144375}\frac{f^{(11)}(z_0)}{11!}h^{11} - \cdots \quad . \tag{2.13}$$

Thus

$$I(f) = R_{1,2}(f) - \frac{608}{7875}\frac{f^{(9)}(z_0)h^9}{9!} - \frac{10688}{144375}\frac{f^{(11)}(z_0)}{11!}h^{11} - \cdots.$$

Degree of precision of the rule $R_{1,2}(f)$:

For k= 0(1)8 we have

$$E_{1,2}((z - z_0)^k) = 0$$

and

$$E_{1,2}((z - z_0)^9) = -\frac{608}{7875}\, h^9 \neq 0.$$

So, the degree of precision of the quadrature rule $R_{1,2}(f)$ is eight.

Following the technique used in the construction of the rule $R_{1,2}(f)$, we multiply the (2.11) by $(\frac{150}{17})$ and (2.10) by $(-\frac{133}{17})$ and then adding the results, we get

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$$I(f) = \left[\frac{150}{17}R_{1,2}(f) - \frac{133}{17}R_3(f)\right] + \left[\frac{150}{17}E_{1,2}(f) - \frac{133}{17}E_3(f)\right]$$
$$= R_{1,2,3}(f) + E_{1,2,3}(f)$$

Where $$R_{1,2,3}(f) = \frac{150}{17}R_{1,2}(f) - \frac{133}{17}R_3(f) \qquad (2.14)$$

is the desired second *composite rule* and the truncation error incurred in approximation by this *composite rule* is

$$E_{1,2,3}(f) = \frac{150}{17}E_{1,2}(f) - \frac{133}{17}E_3(f)\ .$$
$$= -\frac{3392}{363825}\frac{f^{(11)}(z_0)}{11!}h^{11} - \cdots \ .$$

Degree of precision of the composite rules $R_{1,2,3}(f)$.

It is to be noted here that

$$E_{1,2,3}((z-z_0)^k) = \frac{150}{17}E_{1,2}(z-z_0)^k - \frac{133}{17}E_3(z-z_0)^k = 0 \qquad \text{for } k = 0(1)8$$

but $$E_{1,2,3}((z-z_0)^k) = \begin{cases} 0\,; & k = 9,10 \\ -0.009323\, h^{11} \neq 0; & k = 11 \end{cases} \ . \qquad (2.15)$$

This implies that the degree of precision of the *composite rule* $R_{1,2,3}(f)$ is ten.

Next we consider

3. ASYMPTOTIC ERROR ESTIMATES

Now we analyze the errors associated with the *basic rules* $R_1(f)$,$R_2(f)$,$R_3(f)$, and the *composite rules* $R_{1,2}(f)$, and $R_{1,2,3}(f)$ which are prescribed in (2.5), (2.6), (2.7), (2.11) and (2.14) respectively.

Here, we assume that the function f (z) is analytic in the disc:

$$\Omega = \{z: |z-z_0| \leq \rho = r|h|;\ r > 1\}$$

so that the points $z_0, z_0 \pm h$, $z_0 \pm ih$ and $z_0 \pm \alpha h$ are all interior to the disc Ω. Now using Taylor's series expansion of f (z) about z =z_0, we obtain

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n; \qquad (3.1)$$

where $a_n = \frac{f^{(n)}(z_0)}{n!}$ are the Taylor's coefficients.

Since the series given in (3.1) is uniformly convergent in Ω, term by term integration of both sides of (3.1) is possible and it yields

$$I(f) = 2hf'(z_0) + \frac{2}{3}\frac{f^{(3)}(z_0)}{3!}h^3 + \frac{2}{5}\frac{f^{(5)}(z_0)}{5!}h^5 + \frac{2}{7}\frac{f^{(7)}(z_0)}{7!}h^7 + \frac{2}{9}\frac{f^{(9)}(z_0)}{9!}h^9 + \ldots \qquad (3.2)$$

Further using the Taylor's series expansion of each term of $R(f;\alpha)$ about z= z_0 in the disc Ω we obtain after simplification:

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$$R(f;\alpha)=2hf'(z_0)+\frac{2}{3}\frac{f^{(3)}(z_0)}{3!}h^3+\frac{2}{5}\frac{f^{(5)}(z_0)}{5!}h^5+\frac{2(5-12\alpha^2)}{15}\frac{f^{(7)}(z_0)}{7!}h^7+\frac{2(1-4\alpha^4)}{5}\frac{f^{(9)}(z_0)}{9!}h^9+\ldots \quad (3.3)$$

Now substituting $\alpha=\sqrt{\frac{2}{5}}$, $\alpha=\frac{1}{\sqrt{5}}$ and $\sqrt{\frac{5}{21}}$ in succession in the expression given in (3.3), we have

$$R_1(f)2=hf'(z_0)+\frac{2}{3}\frac{f^{(3)}(z_0)}{3!}h^3+\frac{2}{5}\frac{f^{(5)}(z_0)}{5!}h^5+\frac{2}{75}\frac{f^{(7)}(z_0)}{7!}h^7+\frac{18}{125}\frac{f^{(9)}(z_0)}{9!}h^9+\ldots \quad (3.4)$$

$$R_2(f)=2hf'(z_0)+\frac{2}{3}\frac{f^{(3)}(z_0)}{3!}h^3+\frac{2}{5}\frac{f^{(5)}(z_0)}{5!}h^5+\frac{26}{75}\frac{f^{(7)}(z_0)}{7!}h^7+\frac{126}{375}\frac{f^{(9)}(z_0)}{9!}h^9+\ldots \quad (3.5)$$

and

$$R_3(f)=2hf'(z_0)+\frac{2}{3}\frac{f^{(3)}(z_0)}{3!}h^3+\frac{2}{5}\frac{f^{(5)}(z_0)}{5!}h^5+\frac{2}{7}\frac{f^{(7)}(z_0)}{7!}h^7+\frac{682}{2205}\frac{f^{(9)}(z_0)}{9!}h^9+\ldots$$

respectively.

Now from (3.2), (3.4) and (3.2), (3.5) we obtain

$$E_k(f;\alpha)=\begin{cases}0.259048\frac{f^{(7)}(z_0)}{7!}h^7+0.078222\frac{f^{(9)}(z_0)}{9!}h^9+\cdots & \mathrm{k}=1\\ -0.060952\frac{f^{(7)}(z_0)}{7!}h^7-0.113778\frac{f^{(9)}(z_0)}{9!}h^9-\cdots & \mathrm{k}=2,\end{cases}$$

which in turn imply

$$|E_k(f;\alpha)|=O(|h|^7)\,; \qquad k=1,2. \quad (3.6)$$

Now from (2.13) it is also evident that

$$\left|E_{1,2}(f)\right|=O(|h|^9).$$

Also, from (2.15) we have

$$\left|E_{1,2,3}(f)\right|=O(|h|^{11}).$$

ERROR BOUND

The error bounds of four quadrature rules $R_2(f)$, $R_3(f)$, $R_{1,2}(f)$, and $R_{1,2,3}(f)$ constructed in this paper have been obtained here following the technique due to Lether [10]. Since the derivation of error bounds of all the rules constructed are similar to each other, we have derived the error bound of the rule $R_2(f)$ only and it is given in Theorem-3.1. The error bounds of the rules $R_3(f)$, $R_{1,2}(f)$ and $R_{1,2,3}(f)$ are stated in Theorem-3.2 following the technique used in Theorem-3.1.

Further, it is noted here that, the error bound of the quadrature rule $R_1(f)$ given in (2.5) cannot be determined in the same way as it is done for the other four cases, i.e, $R_2(f)$, $R_3(f)$, $R_{1,2}(f)$ and $R_{1,2,3}(f)$ following the technique due to Lether [10] for the reason explained below:

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Since $E_1(f)$ denotes the truncation error in approximation of the integral $I(f)$ by the rule $R_1(f)$,

$$I(f) = R_1(f) + E_1(f)$$

and $E_1(f)$ being a linear operator, we obtain from (3.1) the following:

$$E_1(f) = \sum_{\mu=3}^{\infty} a_{2\mu+1} h^{2\mu+1} E_1(t^{2\mu+1}),$$

by using the transformation $z = z_0 + ht,\ \ t\epsilon[-1,1]$.

Thus, we get

$$E_1(f) = 2\sum_{\mu=3}^{\infty} a_{2\mu+1} \emptyset_1(\mu),$$

where $$\emptyset_1(\mu) = \frac{1}{2\mu+1} - \frac{1}{21}\{(-1)^\mu + 20\left(\frac{2}{5}\right)^\mu\}$$

which is not of one sign for $\mu \geq 3$.

However, its asymptotic error estimate has been given in (3.6). Next, we consider

Error Bound of the rule $R_3(f)$

Theorem 3.1. If $f(z)$ is analytic in a closed disc

$$\Omega = \{z\epsilon C: |z - z_0| \leq \rho = r|h|, r > 1\}$$

Then $$|E_2(f)| \leq 2\text{M}\, e_2(r)$$

where

$$e_2(r) = \left|\ln\left(\frac{r+1}{r-1}\right) - \left(\frac{30r^3 - 26r}{15r^4 - 18r^2 + 3}\right)\right|$$

which tends to zero as $r \to \infty$.

Proof

Here

$$E_2((z - z_0)^k) = I_2((z - z_0)^k) - R_2((z - z_0)^k) = 0 \qquad \text{for k = 0(1)6.}$$

Further E_2(f) being a linear operator, using (3.1) and the transformation $z = z_0 + ht$, $t\epsilon[-1,1]$ we obtain

$$E_2(f) = \sum_{\mu=3}^{\infty} a_{2\mu+1} h^{2\mu+1} E_2(t^{2\mu+1})\ . \tag{3.7}$$

Equation (3.7) can also be written as

$$E_2(f) = \sum_{\mu=3}^{\infty} 2a_{2\mu+1} h^{2\mu+1} \emptyset_2(\mu)$$

where

$$\emptyset_2(\mu) = \frac{1}{2\mu+1} - \frac{1}{3}\left[1 + \frac{5}{6}\left(\frac{1}{5}\right)^\mu\right]$$
$$< 0 \qquad \forall\ \mu \geq 3.$$

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Now using Cauchy inequality [10]

$$|E_2(f)| \le 2M \sum_{\mu=3}^{\infty} \frac{1}{r^{2\mu+1}} |E_2(t^{2\mu+1})|$$
$$= 2Me_2(r).$$

Where

$$M = max_{|z|=\rho}|f(z)|$$

and

$$e_2(r) = \left|E\left(\left(1-\frac{t}{r}\right)^{-1}\right)\right|$$
$$= \left|\ln\left(\frac{r+1}{r-1}\right) - \frac{(30r^3-26r)}{15r^4-18r^2+3}\right|$$

which tends to zero as r→ ∞.

This proves Theorem – 1.

Similarly, now we have

Theorem 3.2.

If $f(z)$ is analytic in a closed disc

$$\Omega = \{z\epsilon C: |z-z_0| \le \rho = r|h|, r > 1\}$$

Then

$$|E_3(f)| \le 2\text{M}e_3(r) \ ,$$
$$|E_{12}(f)| \le 2\text{M}e_{12}(r) \ ,$$
$$|E_{123}(f)| \le 2\text{M}e_{123}(r) \ .$$

Where

$$e_3(r) = \left|\ln\left(\frac{r+1}{r-1}\right) - \frac{(32760r^5-3120r^3-28808r)}{16380r^6-3900r^4-16380r^2+3900}\right| ,$$
$$e_{12}(r) = \left|\ln\left(\frac{r+1}{r-1}\right) - \frac{1}{189}\frac{(3150r^7-1960r^5-1554r^3+772r)}{(25r^8-15r^6-23r^4+15r^2-2)}\right|$$

and

$$e_{123}(r) = \left|\ln\left(\frac{r+1}{r-1}\right) - \frac{1}{51}\left[\frac{(-4282551000r^{13}+2551512600r^{11}+8485451520r^9-5816931120r^7-3870676057r^5+2633150520r^3-332224464r}{(8599500r^{14}-7207200r^{12}-15282540r^{10}+14250600r^8+4766580r^6-6879600r^4+1916460r^2-163800)}\right]\right|$$

which tends to zero as $r \to \infty$.

Proof

This theorem can be established in the same way as it is done in Theorem-3.1, since

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$$E_3(f) = \sum_{\mu=4}^{\infty} 2a_{2\mu+1}h^{2\mu+1}\emptyset_3(\mu)$$

$$E_{12}(f) = \sum_{\mu=4}^{\infty} 2a_{2\mu+1}h^{2\mu+1}\emptyset_{12}(\mu)$$

and

$$E_{123}(f) = \sum_{\mu=5}^{\infty} 2a_{2\mu+1}h^{2\mu+1}\emptyset_{123}(\mu)$$

where

$$\emptyset_3(\mu) = \begin{cases} \frac{1}{2\mu+1} - \left(\frac{17}{120} + \frac{2}{195}\left(\frac{1}{3}\right)^{\mu} - \frac{441}{520}\left(\frac{5}{21}\right)^{\mu}\right) & \forall\, even\ \ \mu \\ \frac{1}{2\mu+1} - \left(\frac{17}{120} - \frac{2}{195}\left(\frac{1}{3}\right)^{\mu} - \frac{441}{520}\left(\frac{5}{21}\right)^{\mu}\right) & \forall\, odd\ \ \mu \end{cases}$$

$$< 0 \qquad \text{for } \mu \geq 4$$

$$\emptyset_{1,2}(\mu) = \begin{cases} \frac{1}{2\mu+1} - \left(\frac{4}{441} + \frac{17}{126} + \frac{20}{21}\left(\frac{2}{5}\right)^{\mu} + \frac{5}{3}\left(\frac{1}{5}\right)^{\mu}\right) & \forall\, even\ \ \mu \\ \frac{1}{2\mu+1} - \left\{\frac{20}{21}\left(\frac{2}{5}\right)^{\mu} + \frac{5}{3}\left(\frac{1}{5}\right)^{\mu} - \left(\frac{4}{441} + \frac{17}{126}\right)\right\} & \forall\, odd\ \ \mu \end{cases}$$

$$< 0 \qquad \text{for } \mu \geq 4.$$

and

$$\emptyset_{1,2,3}(\mu) = \begin{cases} \frac{1}{2\mu+1} - \left(\frac{37}{270} + \frac{10240}{12150}\left(\frac{1}{4}\right)^{\mu} + \frac{49}{2430}\left(\frac{5}{7}\right)^{\mu}\right) & \forall\, even\ \ \mu \\ \frac{1}{2\mu+1} - \left(\frac{37}{270} + \frac{10240}{12150}\left(\frac{1}{4}\right)^{\mu} - \frac{49}{2430}\left(\frac{5}{7}\right)^{\mu}\right) & \forall\, odd\ \ \mu \end{cases}$$

$$< 0 \qquad \text{for } \mu \geq 5.$$

From the asymptotic error estimates of the rules $R_1(f), R_2(f), R_3(f), R_{1,2}(f)$ and $R_{1,2,3}(f)$ constructed in this paper for the numerical evaluation of complex CPV integrals, it also follows that

$$\left|E_{1,2,3}(f)\right| \leq \left|E_{1,2}(f)\right| \leq |E_3(f)| \leq |E_2(f)| \leq |E_1(f)|.$$

This observation is in fact very much noticeable from the numerical integration of the integrals that we have taken and the results of numerical integration are depicted in the tables given in next section.

4. NUMERICAL RESULTS AND DISCUSSION

Under this section, we have considered the following categories of integrals which have been numerically integrated by the set of five quadrature rules

S= $\{R_1(f), R_2(f), R_3(f), R_{1,2}(f), R_{1,2,3}(f)\}$

constructed in formulation of the rule section.

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i) Approximation of the Complex Cauchy Principal Value (CPV) integrals:

Here we have considered the integrals:

$$I_1 = \int_{-i}^{i} \frac{(1+z)e^z}{z} dz\,,\quad I_2 = \int_{\frac{1}{2}(1+i)}^{\frac{3}{2}(1+i)} \frac{\sin z}{z-(1+i)} dz\,, and\quad I_3 = \int_{-i}^{i} \frac{e^z}{z} dz.$$

The results of the numerical integration of the above-mentioned integrals have been depicted in Table-1 given below. The exact values of the integrals are given in the last row of the said Table.

Table 1: Numerical Values of the Complex CPV Integrals

Rule/Integral	$I_1=\int_{-i}^{i} \frac{(1+z)e^z}{z} dz$	$I_2=\int_{\frac{1}{2}(1+i)}^{\frac{3}{2}(1+i)} \frac{\sin z}{z-(1+i)} dz$	$I_3=\int_{-i}^{i} \frac{e^z}{z} dz$
$R_1(f;\alpha)$	3.575 517 2 i	1.817 588 2−0.205 730 9 i	1.892 217 3 i
$R_2(f;\alpha)$	3.575 014 4 i	1.817 588 8−0.205 723 7 i	1.892 154 4 i
$R_3(f;\alpha)$	3.575 110 5 i	1.817 588 8−0.205 725 5 i	1.892 166 4 i
$R_{1,2}(f;\alpha)$	3.575 110 2 i	1.817 588 7−0.205 725 2 i	1.892 166 4 i
$R_{1,2,3}(f;\alpha)$	3.575 108 1 i	1.817 588 7−0.205 725 1 i	1.892 166 1 i
Exact value	3.575 108 1 i	1.817 588 7−0.205 725 1 i	1.892 166 1 i

From the approximate values of the integrals reported in Table-1, it is observed that the approximations steadily improve from a minimum of three decimal places (in case of $R_1(f)$) to a maximum of seven decimal places (in case of $R_{1,2,3}(f)$) of accuracy.

Approximations of each of the integrals show a trend of convergence from which the user can confidently accept the final approximation correct to certain decimal places as the true value of the integral, which is a positive advantage when an unknown integral is numerically integrated by the sequence of rules of increasing precision given in set S.

ii) Numerical integration of Real Cauchy Principal Value (CPV) integrals:

Through numerical integrations, it is shown that the real CPV integrals can also be evaluated with approximately almost eight decimal accuracies by the set of rules S given in the formulation of the rule section. For this purpose, we have chosen the integrals:

$$J_1 = \int_{-1}^{1} \frac{e^x}{x} dx,\quad J_2 = \int_{-1}^{1} \frac{\sin x}{x} dx,\quad J_3 = \int_{-1}^{1} \frac{\cos x}{x} dx$$

The approximations obtained by numerical integration of the integrals J_k; k=1, 2, 3 are depicted in Table-2 along with their exact values noted in the last row of the table.

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Table-2: Numerical values of the Real CPV Integrals

Rule/Integral	$J_1=\int_{-1}^{1}\frac{e^x}{x}dx$	$J_2=\int_{-1}^{1}\frac{sinx}{x}dx$	$J_3=\int_{-1}^{1}\frac{cosx}{x}dx$
$R_1(f)$	2.114 450 13	1.892 217 33	0.0
$R_2(f)$	2.114 514 16	1.892 154 36	0.0
$R_3(f)$	2.114 501 99	1.892 166 38	0.0
$R_{1,2}(f)$	2.114 501 97	1.892 166 35	0.0
$R_{1,2,3}(f)$	2.114 501 75	1.892 166 14	0.0
Exact value	2.114 501 75	1.892 166 14	0.0

iii) Evaluation of Complex/Real definite integrals:

Integration of analytic functions over a line segment L in the complex plane C can also be evaluated approximately by the same set of rules S.

For instance, if we desire to determine the approximations of the integrals

$$K = \int_L f(z)dz$$

where L is a line segment joining the points from $z_0 - h$ to $z_0 + h$ in the complex plane C, we rewrite the same integral in the form

$$K = \int_{z_0-h}^{z_0+h} \frac{\emptyset(z)}{z} dz; \ \emptyset(z) = z f(z)$$

and then approximate it by the rules under consideration choosing $\emptyset(z)$ instead of f(z) in the quadrature rules.

Following this simple technique, the definite integrals

$$k_1 = \int_{-i}^{i} e^z dz, \quad k_2 = \int_{-i/2}^{i/2} cosz\, dz, \quad k_3 = \int_{-\frac{(1+i)}{\sqrt{2}}}^{\frac{(1+i)}{\sqrt{2}}} ze^z dz$$

have been numerically integrated and their approximate values along with their respective exact values are shown in Table-(4.3).

Table 3: Numerical Values of the Complex Definite Integrals

Rule/Integral	$k_1 = \int_{-i}^{i} e^z dz$	$k_2 = \int_{-i/2}^{i/2} cosz\, dz$	$k_3 = \int_{-\frac{(1+i)}{\sqrt{2}}}^{\frac{(1+i)}{\sqrt{2}}} ze^z dz$
$R_1(f)$	1.683 299 89 i	1.042 187 80 i	−0.518 367 5+0.424 126 9 i
$R_2(f)$	1.682 860 09 i	1.042 191 28 i	−0.516 455 7+0.422 269 1 i
$R_3(f)$	1.682 944 11 i	1.042 190 62 i	−0.516 818 5+0.422 624 4 i
$R_{1,2}(f)$	1.682 943 86 i	1.042 190 61 i	−0.516 819 9+0.422 622 9 i
$R_{1,2,3}(f)$	1.682 941 97 i	1.042 190 61 i	−0.516 830 6+0.422 612 0 i
Exact value	1.682 941 97 i	1.042 190 61 i	−0.516 830 5+0.422 612 0 i

The error function $erf\, x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ has been numerically evaluated for x=1 and the approximations are shown in Table-4.

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Table 4: Numerical values of the error function

Rule/Integral	$\frac{\sqrt{\pi}}{2}\, erf1 = \int_0^1 e^{-t^2} dt$
$R_1(f)$	**0.746 758 31**
$R_2(f)$	**0.746 836 60**
$R_3(f)$	**0.746 821 37**
$R_{1,2}(f)$	**0.746 821 69**
$R_{1,2,3}(f)$	**0.746 824 15**
Exact value	**0.746 824 13**

iv) Approximations of some Hyper singular integrals:

The Hyper singular integral of the type

$$H \int_a^b \frac{f(x)}{(x-s)^2} dx,$$

appears in the formulation of the crack problems in linear elastic fracture mechanics and many other applied sciences such as acoustic, fluid mechanics, elasticity, and so on. Here, we intend to illustrate the numerical integration of Hyper singular integrals of the type

$$\mathrm{I}_H = H \int_{-1}^{1} \frac{f(x)}{x^2} dx,$$

by the rules constructed for approximation of complex CPV integrals in formulation of the rule section. For this, we write the integral as

$$\mathrm{I}_H = H \int_{-1}^{1} \frac{f(z)}{z^2} dz \qquad (4.1)$$

To accomplish the approximation, we rewrite I_H in (4.1) as

$$\mathrm{I}_H = P \int_{-1}^{1} \left(\frac{f(z)-f(0)}{z}\right)\frac{1}{z} dz + f(0)\, H \int_{-1}^{1} \frac{1}{z^2} \mathrm{dz}$$
$$= H' + f(0)\, H \int_{-1}^{1} \frac{1}{z^2} \mathrm{dz} \qquad (4.2)$$

The first integral H' on the righthand side of (4.2) is a complex CPV integral which can be approximated by the set of rules given in the set S meant for approximation of complex CPV integral. The second integral exists as a Hyper singular integral and its value is easily seen to be -2. Following this technique, the Hyper singular integrals such as

$$H_1 = H \int_{-1}^{1} \frac{cosx}{x^2} dx, \quad H_2 = H \int_{-1}^{1} \frac{e^{ix}}{x^2} dx \text{, and } H_3 = H \int_{-1}^{1} \frac{e^{x}}{x^2} dx,$$

have been numerically evaluated. The approximations of H' for each such integral along with the corresponding approximation values of the integrals $(H' - 2f(0))$ are depicted in Table-5 to Table-7 respectively.

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Table 5: Numerical Values of Hyper singular Integral H_1

Rule/Integral	H_1'	$H_1 = H_1' - 2f(0)$
$R_1(f)$	-0.972 777 16	-2.972 777 16
$R_2(f)$	-0.972 769 27	-2.972 769 27
$R_3(f)$	-0.972 770 78	-2.972 770 78
$R_{1,2}(f)$	-0.972 770 77	-2.972 770 77
$R_{1,2,3}(f)$	-0.972 770 75	-2.972 770 75
Exact value	*********	-2.972 770 75

Table 6: Numerical values of the Hyper singular Integral H_2

Rule/Integral	H_3'	$H_3 = H_3' - 2f(0)$
$R_1(f)$	1.028 334 03	-0.971 665 97
$R_2(f)$	1.028 342 02	-0.971 657 98
$R_3(f)$	1.028 340 51	-0.971 659 49
$R_{1,2}(f)$	1.028 340 50	-0.971 659 50
$R_{1,2,3}(f)$	1.028 340 48	-0.971 659 52
Exact value	**********	-0.971 659 52

Table7: Numerical values of Hypersingular Integral H_3

Rule/Integral	H_2'	$H_2 = H_2' - 2f(0)$
$R_1(f)$	-0.972 770 16	-2.972 770 16
$R_2(f)$	-0.972 769 27	-2.972 769 27
$R_3(f)$	-0.972 770 78	-2.972 770 78
$R_{1,2}(f)$	-0.972 770 77	-2.972 770 77
$R_{1,2,3}(f)$	-0.972 770 75	-2.972 770 75
Exact value	**********	-2.972 770 75

CONCLUSION

All the integrals under numerical verification have been numerically integrated by the sequence of quadrature rules of increasing precision; the precision increasing from six to ten. As a result, the approximate values of each of the integrals constantly improve and converge to the exact values and ultimately the values agree to eight decimal places.

Once the functions at the nodes present in the basic rules $R_1(f)$, $R_2(f)$ and $R_3(f)$ have been evaluated, the integral given in (1.1) can be numerically integrated by the composite rules $R_{1,2}(f)$ and $R_{1,2,3}(f)$ by a few arithmetic operations like addition and multiplication without incurring any additional truncation error. Also, the round of error in the last two approximations i.e. by the rules $R_{1,2}(f)$ and $R_{1,2,3}(f)$ will be negligible.

Further between the rules $R_1(f)$ and $R_2(f)$, which are the same precision six, the rule $R_2(f)$ integrates more accurately than $R_1(f)$ since

$$|E_1(f)| = 0.259048|h|^7$$

and

$$|E_2(f)| = 0.060952|h|^7.$$

This fact is also reflected in the approximation of integrals by the rules $R_1(f)$ and $R_2(f)$ reported in the Tables of approximation in the numerical section.

The same fact is also noted in the case of the rules $R_3(f)$ and $R_{1,2}(f)$; $R_{1,2}(f)$ integrates more accurately than $R_3(f)$ as

$$|E_3(f)| = 0.087075|h|^9$$

and

$$\left|E_{1,2}(f)\right| = 0.077206|h|^9 .$$

Over and above, the rule $R_{1,2,3}(f)$, being a rule of precision ten, produces the most accurate approximation among all the rules constructed in this paper.

As it is demonstrated through examples, Hyper singular integrals otherwise known as Hadamard finite part integrals can also be numerically integrated by the rules constructed in this paper to eight decimal accuracy.

Acknowledgements

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

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Journal of Tianjin University Science and Technology
ISSN (Online):0493-2137
E-Publication: Online Open Access
Vol: 56 Issue: 07: 2023
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