Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
Journal of Tianjin University Science and Technology
ISSN (Online):0493-2137
E-Publication: Online Open Access
Vol: 58 Issue: 11:2025
DOI: 10.5281/zenodo.17656933

PROPER d- LUCKY LABELING ON SPECIAL GRAPHS

S. SUNDARRAJ
S.I.V.E.T. College, Gowrivakkam, Chennai. Email. sjsundar89@gmail.com

S. BALA
S.I.V.E.T. College, Gowrivakkam, Chennai. Email: yesbala75@gmail.com

V.SUGANYA
S.I.V.E.T. College, Gowrivakkam, Chennai. Email: sugan4kavi@gmail.com

Abstract

In this paper we investigate the existence of Proper d-lucky labeling for some special graphs.

Keywords: Proper d-Lucky Labeling, Twig, Barycentric, Cycle, Kite.

1. INTRODUCTION

The concept of graph labeling was introduced by Rosa in 1967 [8]. Mirka Miller et.al., [7] introduced the concept of **d-lucky labeling**. Let $l: R(G) \to \mathbb{N}$ be the vertex labels of G which is assigned by positive integers. Define $c(u) = d(u) + \sum_{r \in N(u)} l\,(u)$, [where $d(u)$ is the degree of u and $N(u)$ is the open neighbourhood of u].

A Labeling l is expound to be D-lucky if $c(u)$ and $c(r)$ are not equal. For every couple of adjacent vertices u and r in G.

Indicate the D-lucky number of a graph G by $\eta_{dl}(G)$, which is the smallest positive k in order that G has a D-lucky labeling with $\{1,2, \ldots., k\}$ as the set of labels. Esakkiammal et.al., [6] introduced the concept of **Proper d-lucky Labeling**.

A D-lucky labeling is called proper if $l(u) \neq l(r)$ for every adjacent vertex u and r. The proper D-lucky number of a graph is the least positive integ k such that G has a Proper D-lucky labeling with $\{1,2, \ldots., k\}$ as the set of labels and is denoted by $\eta_{pdl}(G)$. Bala et.al., introduced the concept of Triplicate **Graph of Cycle** C_n.

Let $V = \{v_i \cup v_i' \cup v_i''/1 \leq i \leq n\}$ and edge set $E = \{v_{i+1}'v_i \cup v_{i+1}v_i' \cup v_i'v_{i+1}'' \cup v_i''v_{i+1}'/1 \leq i \leq n-\ 1\} \cup \{v_nv_n' \cup v_n'v_1 \cup v_n'v_1'' \cup v_n''v_1'\ /1 \leq i \leq n\}$. Clearly, $TG(C_n)$ has $3n$ vertices and $4n$ edges.

Bala et.al., [2] introduced the concept of **Triplicate Graph of kite** $TG(\text{kite})_n$. Let $V = \{v_j \cup v_j' \cup v_j'' \cup u_i \cup u_i' \cup u_i''/1 \leq i \leq n\}$ and edge set $E = \{v_{i+1}'v_i \cup v_{i+1}v_j' \cup v_i'v_{i+1}'' \cup v_i''v_{i+1}'/1 \leq i \leq n-1\} \cup \{v_nv_1' \cup v_n'v_1 \cup v_n'v_1'' \cup v_n''v_1'\} \cup \{v_1'u_1 \cup v_1u_1' \cup v_1'u_1'' \cup v_1''u_1'\} \cup \{u_iu_{i+1}' \cup u_i'u_{i+1} \cup u_i'u_{i+1}'' \cup u_i''u_{i+!}'\ /1 \leq i \leq n-1\}$.

Clearly, $TG(\text{kite})_n$has $6n$ vertices and $8n$ edges. Bala et.al., [1] introduced the concept of **The Barycentric Extended Duplicate Graph of Twig** $EDG(T_n)$. Let $V = \{v_j \cup v_j'/1 \leq i \leq n+2\} \cup \{b_i \cup r_i/1 \leq i \leq n+1\}\,\{b_i' \cup r_i' \cup b_i'' \cup r_i'' \cup u_i \cup u_i' \cup w_i \cup w_i'\ /$

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
Journal of Tianjin University Science and Technology
ISSN (Online):0493-2137
E-Publication: Online Open Access
Vol: 58 Issue: 11:2025
DOI: 10.5281/zenodo.17656933

$2 \leq i \leq n+1\} \cup \{s_1\}$ and edge set $E = \{v_i b_i \cup r_i v_{i+1} \cup v_i' r_i \cup b_i v_{i+1}' / 1 \leq i \leq n+1\} \cup \{v_i b_i'' \cup v_i' r_i'' \cup v_i' b_i' \cup v_i' r_i' \cup b_i' u_i \cup r_i' w_i \cup b_i'' u_i' \cup r_i'' w_i' / 2 \leq i \leq n+1\} \cup \{v_2 s_1 \cup v_2' s_1\}$.
Clearly, $EDG(T_n)$ has $12n+7$ vertices and $12n+6$ edges.

2. MAIN RESULT

In this section we investigate the existence of Proper D-Lucky Labeling on Triplicate Graph of cycle, Triplicate Graph of kite and Barycentric Extended Duplicate Graph of Twig.

THEOREM: 2.1

Triplicate Graph of cycle $TG(C_n)$ admits proper d-Lucky labeling with $\eta_{dl}(TG(C_n)) = 2$.

Proof:

The Triplicate Graph of Cycle C_n, denoted by $TG(C_n)$ has the vertex set $V = \{v_i \cup v_i' \cup v_i'' / 1 \leq i \leq n\}$ and edge set $E = \{v_{i+1}' v_i \cup v_{i+1} v_i' \cup v_i' v_{i+1}'' \cup v_i'' v_{i+1}' / 1 \leq i \leq n-1\} \cup \{v_n v_n' \cup v_n' v_1 \cup v_n' v_1'' \cup v_n'' v_1'\}$.

Clearly, $TG(C_n)$ has $3n$ vertices and $4n$ edges.

To prove $TG(C_n)$ is Proper d-lucky, define the function $l: V(G) \to \mathbb{N}$ to label the vertices as follows:

For, $1 \leq i \leq n$

$$l(v_i) = l(v_i'') = 1,\ l(v_i') = 2$$

Clearly, it has $l(v_i) \neq l(v_j)$ for any two adjacent vertices of $TG(C_n)$.

From the structure of the $TG(C_n)$, it is clear that the degrees of the vertices are as follows:

For, $1 \leq i \leq n$

$$\mathrm{d}(v_i) = d(v_i'') = 2,\ d(v_i') = 4$$

Using the relation $c(v_i) = d(v_j) + \sum_{v_j \in N(v_i)} l(v_i)$ we obtain the d-lucky number as follows

For, $1 \leq i \leq n$

$$\mathrm{c}(v_i) = c(v_i'') = 6,\ c(v_i') = 8$$

Clearly,

(i) $c(v_n) \neq c(v_n')$, $c(v_n') \neq c(v_1)$, $c(v_n') \neq c(v_1'')$, $c(v_n'') \neq c(v_1')$

(ii) For, $1 \leq i \leq n-1$

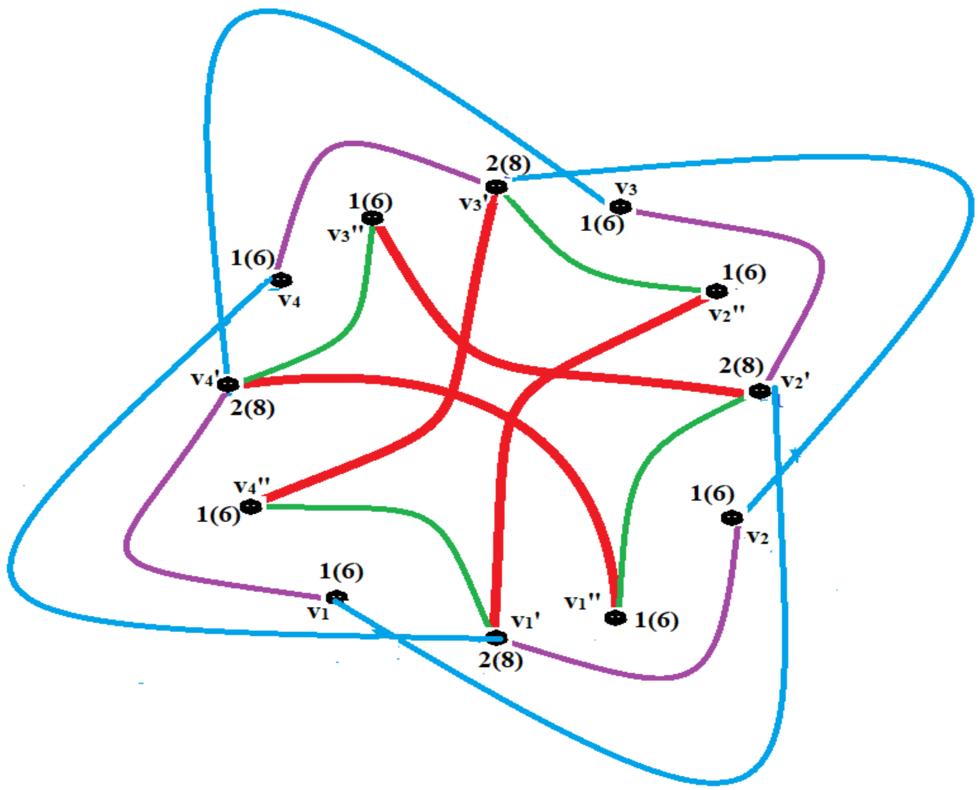
$$c(v_{i+1}') \neq c(v_1),\ c(v_{i+1}) \neq c(v_1'),\ c(v_i') \neq c(v_{i+1}''),\ c(v_i'') \neq c(v_{i+1}')$$

Since any two adjacent vertices of $TG(C_n)$ are not equal. $TG(C_n)$ admits proper d-lucky labeling with $\eta_{dl}(TG(C_n)) = 2$.

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
Journal of Tianjin University Science and Technology
ISSN (Online):0493-2137
E-Publication: Online Open Access
Vol: 58 Issue: 11:2025
DOI: 10.5281/zenodo.17656933

EXAMPLE: 2.1

Proper d-lucky labeling $TG(C_n)$ is shown in the figure.



PROPER D- LUCKY LABELING

THEOREM: 2.2

Triplicate Graph of kite $TG(\text{kite})_n$ admits proper d-Lucky labeling with $\eta_{dl}(TG(\text{kite})_n\,) = 2$.

Proof:

The Triplicate Graph of kite $(\text{kite})_n$. denoted by TG(kite)n has the vertex set $V = \left\{v_j \cup v'_j \cup v''_j \cup u_i \cup u'_i \cup u''_i / 1 \le i \le n\right\}$ and edge set $E = \left\{v'_{i+1}v_i \cup v_{i+1}v'_j \cup v'_i v''_{i+1} \cup v''_i v'_{i+1} / 1 \le i \le n-1\right\} \cup \{v_n v'_1 \cup v'_n v_1 \cup v'_n v''_1 \cup v''_n v'_1\} \cup \{v'_1 u_1 \cup v_1 u'_1 \cup v'_1 u''_1 \cup v''_1 u'_1\} \cup \{u_i u'_{i+1} \cup u'_i u_{i+1} \cup u'_i u''_{i+1} \cup u''_i u'_{i+!} / 1 \le i \le n-1\}$.

Clearly, $TG(\text{kite})_n$has $6n$ vertices and $8n$ edges.

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
Journal of Tianjin University Science and Technology
ISSN (Online):0493-2137
E-Publication: Online Open Access
Vol: 58 Issue: 11:2025
DOI: 10.5281/zenodo.17656933

To prove $TG(\text{kite})_n$ is Proper D-lucky, define the function $l: V(G) \to \mathbb{N}$ to label the vertices as follows:

For, $1 \leq i \leq n$

$$l\,(v_i) = l\,(v_i'') = l\,(u_i) = l\,(u_i'') = 1,\; l\,(v_i') = l\,(u_i') = 2,$$

Clearly, it has $l(v_i) \neq l(v_j)$ for any two adjacent vertices of $TG(\text{kite})_n$.

From the structure of the $TG(\text{kite})_n$, it is clear that the degrees of the vertices are as follows:

(i) $\mathrm{d}\,(v_1) = d\,(v_1'') = 3,\; d\,(v_1') = 6$

(ii) For, $2 \leq i \leq n$

$$\mathrm{d}\,(v_i) = d\,(v_i'') = 2,\; d\,(v_i') = 4$$

(iii) For, $1 \leq i \leq n-1$

$$\mathrm{d}\,(u_i) = d\,(u_i'') = 2,\; d\,(u_i') = 4$$

(iv) $\mathrm{d}\,(u_n) = d\,(u_n'') = 1,\; d\,(u_n') = 2$

Using the relation $c(v_i) = d\big(v_j\big) + \sum_{v_j \in N(v_i)} l\,(v_i)$ we obtain the d-lucky number as follows

(i) $\mathrm{c}\,(v_1) = c\,(v_1'') = 9,\; c\,(v_1') = 12,\;\; \mathrm{c}\,(u_n) = c\,(u_n'') = 3,\; c\,(u_n') = 4$

(ii) For, $2 \leq i \leq n$

$$\mathrm{c}\,(v_i) = c\,(v_i'') = 6,\; c\,(v_i') = 8$$

(iii) For, $1 \leq i \leq n-1$

$$\mathrm{c}\,(u_i) = c\,(u_i'') = 6,\; c\,(u_i') = 8$$

Clearly,

(i) $c(v_n) \neq c(v_n'),\; c(v_n') \neq c(v_1),\; c(v_n') \neq c(v_1''),\; c(v_n'') \neq c(v_1')$

(ii) For, $1 \leq i \leq n-1$

$$c(v_{i+1}') \neq c(v_1),\; c(v_{i+1}) \neq c(v_1'),\; c(v_i') \neq c(v_{i+1}''),\; c(v_i'') \neq c(v_{i+1}')$$

$$c(u_1') \neq c(u_{i+1}'),\; c(u_i') \neq c(u_{i+1}),\; c(u_i') \neq c(u_{i+1}''),\; c(u_i'') \neq c(u_{i+!}')$$

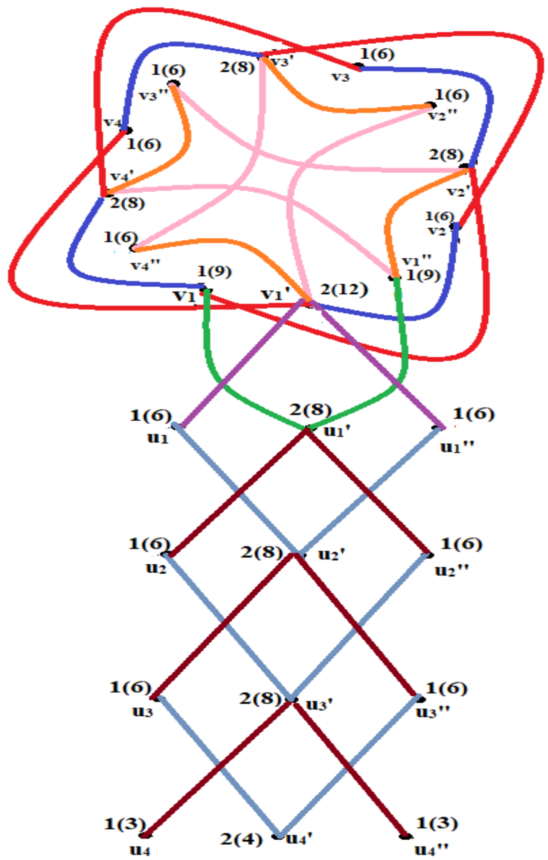
(iii) $c(v_1') \neq c(u_1),\; c(v_1) \neq c(u_1'),\; c(v_1') \neq c(u_1''),\; c(v_1'') \neq c(u_1')$

Since any two adjacent vertices of $TG(\text{kite})_n$ are not equal. $TG(\text{kite})_n$ admits proper d-lucky labeling with $\eta_{dl}(TG(\text{kite})_n\,) = 2$.

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
Journal of Tianjin University Science and Technology
ISSN (Online):0493-2137
E-Publication: Online Open Access
Vol: 58 Issue: 11:2025
DOI: 10.5281/zenodo.17656933

EXAMPLE: 2.2

Proper d-lucky labeling $TG(\text{kite})_4$is shown in the figure.



PROPER D- LUCKY LABELING

THEOREM: 3.3

The Barycentric Extended Duplicate Graph of Twig $EDG(T_n)$ admits proper d-Lucky labeling with $\eta_{dl}(EDG(T_n)\,) = 2$.

Proof:

The Barycentric Extended Duplicate Graph of Twig $EDG(T_n)$. denoted by EDG(T_n) has the vertex set $V = \{v_j \cup v_j'/1 \le i \le n+2\} \cup \{b_i \cup r_i/1 \le i \le n+1\}\{b_i' \cup r_i' \cup b_i'' \cup r_i'' \cup u_i \cup u_i' \cup w_i \cup w_i'/2 \le i \le n+1\} \cup \{s_1\}$ and edge set $E = \{v_i b_i \cup r_i v_{i+1} \cup v_i' r_i \cup b_i v_{i+1}'/1 \le i \le n+1\} \cup \{v_i b_i'' \cup v_i{}' r_i'' \cup v_i' b_i' \cup v_i' r_i' \cup b_i' u_i \cup r_i' w_i \cup b_i'' u_i' \cup r_i'' w_i'/2 \le i \le n+1\} \cup \{v_2 s_1 \cup v_2' s_1\}$.

Clearly, $EDG(T_n)$ has $12n + 7$ vertices and $12n + 6$ edges.

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
Journal of Tianjin University Science and Technology
ISSN (Online):0493-2137
E-Publication: Online Open Access
Vol: 58 Issue: 11:2025
DOI: 10.5281/zenodo.17656933

To prove $EDG(T_n)$ is Proper d-lucky, define the function $l: V(G) \to \mathbb{N}$ to label the vertices as follows:

(i) For, $1 \le i \le n+2$

$$l\,(v_i) = l\,(v_i') = 1$$

(ii) For, $2 \le i \le n+1$

$$l\,(b_i') = l\,(b_i'') = l\,(r_i') = l\,(r_i'') = 2$$

(iii) For, $1 \le i \le n+1$

$$l\,(b_i) = l\,(r_i) = 2$$

(iv) For, $2 \le i \le n+1$

$$l\,(u_i) = l\,(u_i') = l\,(w_i) = l\,(w_i') = 1$$

(v) $l\,(s_1) = 2$

Clearly, it has $l(v_i) \ne l(v_j)$ for any two adjacent vertices of (T_n) .

From the structure of the (T_n) , it is clear that the degrees of the vertices are as follows:

(i) $\mathrm{d}\,(v_1) = \mathrm{d}\,(v_{n+2}) = d\,(v_1') = d\,(v_{n+2}') = 1, d\,(v_2) = d\,(v_2') = 5, d\,(s_1) = 2$

(ii) For, $3 \le i \le n+1$

$$\mathrm{d}\,(v_i) = d\,(v_i') = 4$$

(iii) For, $2 \le i \le n+1$

$$d\,(b_i) = d\,(b_i'') = d\,(r_i) = d\,(r_i'') = 2,$$

$$d\,(u_i) = d\,(u_i') = d\,(w_i) = d\,(w_i') = 1$$

(iv) For, $1 \le i \le n+1$

$$d\,(b_i) = d\,(r_i) = 2$$

Using the relation $c(v_i) = d(v_j) + \sum_{v_j \in N(v_i)} l\,(v_i)$ we obtain the d-lucky number as follows

(i) $\mathrm{c}\,(v_1) = \mathrm{c}\,(v_{n+2}) = c\,(v_1') = c\,(v_{n+2}') = 3, c\,(v_2) = c\,(v_2') = 15, c\,(s_1) = 4$

(ii) For, $3 \le i \le n+1$

$$\mathrm{c}\,(v_i) = c\,(v_i') = 12$$

(iii) For, $2 \le i \le n+1$

$$c\,(b_i') = c\,(b_i'') = c\,(r_i') = c\,(r_i'') = 4,$$

$$c\,(u_i) = c\,(u_i') = c\,(w_i) = c\,(w_i') = 3$$

(iv) For, $1 \le i \le n+1$

$$c\,(b_i) = c\,(r_i) = 4$$

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
Journal of Tianjin University Science and Technology
ISSN (Online):0493-2137
E-Publication: Online Open Access
Vol: 58 Issue: 11:2025
DOI: 10.5281/zenodo.17656933

Clearly,

(i) $c(v_2) \neq c(s_1)$, $c(v_2') \neq c(s_1)$

(ii) For, $1 \leq i \leq n+1$

$$c(v_i) \neq c(b_i),\ c(r_i) \neq c(v_{i+1}),\ c(v_i') \neq c(r_i),\ c(b_i) \neq c(v_{i+1}')$$

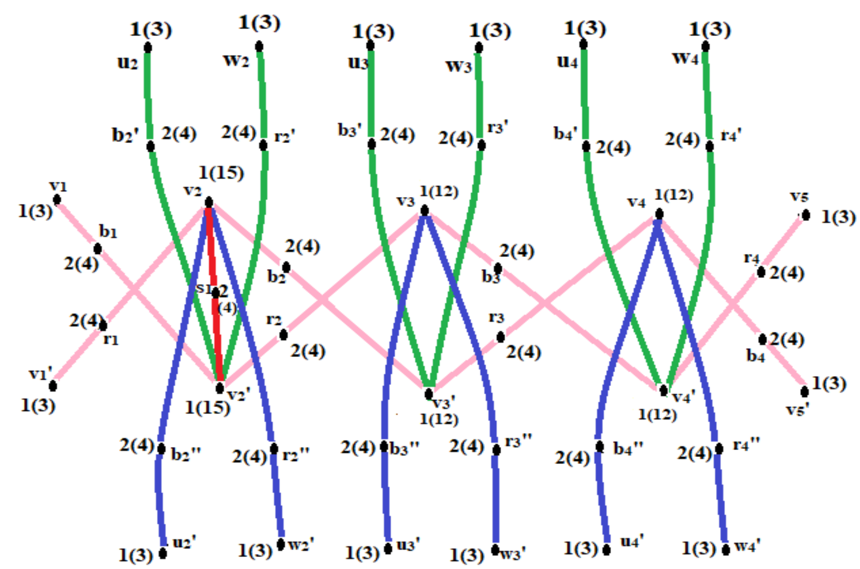
(iii) For, $2 \leq i \leq n+1$

$$c(v_i) \neq c(b_i''),\ c(v_i) \neq c(r_i''),\ c(v_i') \neq c(b_i'),\ c(v_i') \neq c(r_i'),\ c(b_i') \neq c(u_i),\ c(r_i') \neq c(w_i),$$
$$c(b_i'') \neq c(u_i'),\ c(r_i'') \neq c(w_i')$$

Since any two adjacent vertices of $EDG(T_n)$ are not equal. $EDG(T_n)$ admits proper d-lucky labeling with $\eta_{dl}(\ EDG(T_n)\ ) = 2$.

EXAMPLE: 3.3

Proper d-lucky labeling $EDG(T_n)$is shown in the figure.



PROPER D- LUCKY LABELING

CONCLUSION

In this paper, we have confirmed the existence of Proper d-Lucky Labeling on Triplicate Graph of Cycle, Triplicate Graph of Kite and Barycentric Extended Duplicate Graph of Twig.

Tianjin Daxue Xuebao (Ziran Kexue yu Gongcheng Jishu Ban)/
Journal of Tianjin University Science and Technology
ISSN (Online):0493-2137
E-Publication: Online Open Access
Vol: 58 Issue: 11:2025
DOI: 10.5281/zenodo.17656933

References

1) Bala.S, Sundarraj.S, Thirusangu.K, Some Cordial labeling for Barycentric twig family graph, AIP Conference proceedings 2261,030100(2020)05 October 2020, page number: 030100-1 to 030100-8, ISSN: 1551-7616.

2) Bala.S, Sundarraj.S, Thirusangu.K, Cordial Labeling for Some Special Kite Graph, International conference on Research Trends in Mathematics-2020(ICRTM_2020) Proceedings of International conference on Research Trends in Mathematics, page number:219-224, ISBN: 978-93-92811-09-8,

3) Bala.S, Sundarraj.S, Thirusangu.K, Cordial Labelings for Twig related Graph, Journal of Applied Science and Computations, Volume VI, Issue IV, April 2019, page number: 234-241.

4) Bala.S, Suganya V, Thirusangu k, D-Lucky Labeling of Some Special Graphs, Purakala, V0l 32 Issue 1, pg.no315-319,2023. ISSN :0971-2143.

5) Bala.S, Suganya.V, Thirusangu. K, D-lucky labeling for some extended triplicate graph, international journal of Research and analytical reviews, IJRAR, Volume 11, Issue 1, E-ISSN 2348-1269, P- ISSN 2349-5138, March 2024.

6) Esakkiammal E, Thirusangu K, Seethalakshmi S, Proper d-lucky labeling on arbitrary supersubdivision of some graphs, Journal of Applied Science and Computations. Vol.vi (1), (2019), pp.575-602.

7) Mirka Miller, Indra Rajasingh. D, Ahima Emilet, Azubha Jemilet.D, d-Lucky Labeling of Graphs, *Procedia Computer Science*, 57 (2015), pp 766-771.

8) Rosa.A, On certain valuations of the vertices of a graph, Theory of graphs (Internat.Symposium, Rome, July 1996), Gordon and Breach, N.Y. and Dunod Paris (1967), 349-355.